

Examples:

$$\frac{\begin{array}{c} (1) \phi + * : \square \text{ (soul)} \\ (2) L : * + L : * \text{ (soul)} \end{array}}{(4) L : *, x : L + L : * \text{ (weak)}}$$

$$\frac{\begin{array}{c} (1) \phi + * : \square \text{ (soul)} \\ (2) L : * + L : * \text{ (soul)} \end{array}}{(5) L : * + * : \square \text{ (weak)}}$$

$$(3) \phi + * : \square \text{ (soul)}$$

$$\frac{\begin{array}{c} (1) \phi + * : \square \text{ (soul)} \\ (2) L : * + L : * \text{ (soul)} \end{array}}{(6) L : *, B : * + L : * \text{ (weak)}}$$

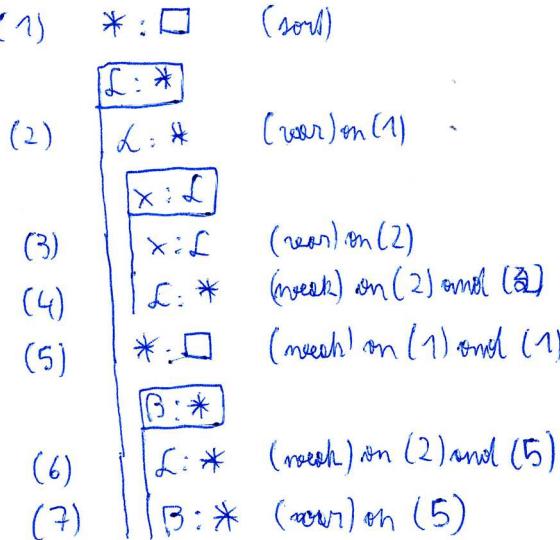
$$\frac{\begin{array}{c} (1) \phi + * : \square \text{ (soul)} \\ (2) L : * + L : * \text{ (soul)} \end{array}}{(7) L : *, B : * + B : * \text{ (soul)}}$$

$$(5) L : * + * : \square \text{ (weak)}$$

$$\frac{\begin{array}{c} (1) \phi + * : \square \text{ (soul)} \\ (2) L : * + L : * \text{ (soul)} \end{array}}{(7) L : *, B : * + B : * \text{ (soul)}}$$

$$(3) L : * + * : \square \text{ (weak)}$$

Tree formed:



The formation rule (form)

$$\text{(form)} \frac{P \vdash A : s \quad \Gamma \vdash B : s}{\Gamma \vdash A \rightarrow B : s}$$

Example:

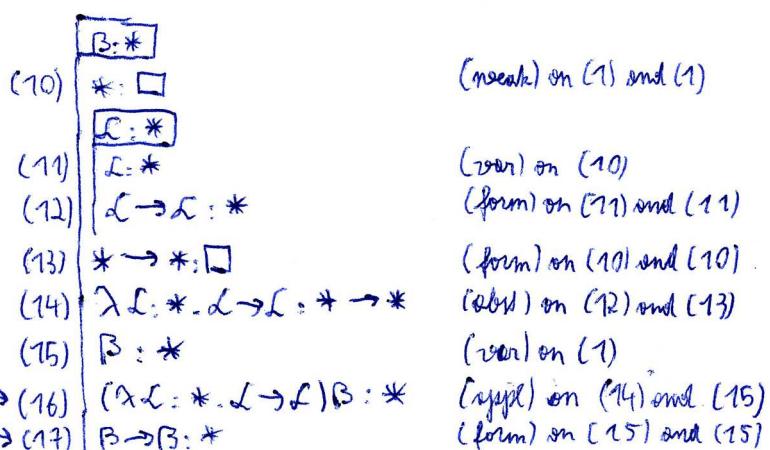
$$\begin{array}{c}
(8) \vdash L \rightarrow B : * \text{ (form) on (6) and (7)} \\
(9) * \rightarrow * : \square \text{ (form) on (5) and (5)}
\end{array}$$

The application and abstraction rules

$$\text{(appl)} \frac{\Gamma \vdash M : A \rightarrow B, \Gamma \vdash N : A}{\Gamma \vdash MN : B}$$

$$\text{(sub)} \frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash A \rightarrow B : s}{\Gamma \vdash \lambda x : A. M : A \rightarrow B}$$

$$(\lambda L : *, L \rightarrow L) B \rightarrow B \rightarrow B$$



Shortened derivation

Silently execute (var), (var), (metab.) and (form) rules; second premise of (subst) rule

(a)	$B : *$	
(b)	$L : *$	necessary for (14)
(12)	$L \rightarrow L : *$	(form) on (b) and (b)
(14)	$\lambda L : *. L \rightarrow L : * \rightarrow *$	(subst) in (12) ← first premise only
(16)	$(\lambda L : *. L \rightarrow L) B : *$	(apply) on (14) and (a)

The conversion rule

Problem:  $(\lambda L : *. L \rightarrow L) B \rightarrow_B \beta \rightarrow \beta$

$$B : *, x : (\lambda L : *. L \rightarrow L) B \vdash x : (\lambda L : *. L \rightarrow L) B \quad (\text{var}) \text{ on (16)}$$

naturally, we also want

$$B : *, x : (\lambda L : A. L \rightarrow L) B \vdash x : B \rightarrow \beta$$

but this cannot be derived so far!

Solution: (conv) 
$$\frac{\Gamma \vdash A : B \quad \Gamma \vdash B' : S}{\Gamma \vdash A : B'} \quad \text{if } B =_B B'$$

thus  $B$  is already well-formed  
but  $B'$  may be not, even if  $B =_B B'$ :  $B \rightarrow \gamma =_B (\lambda L : *. B \rightarrow \gamma) M$   
still for every term  $M$ , but  
 $(\lambda L : *. B \rightarrow \gamma) M$  need not be  
well-formed (if  $M$  is not)

Example:

(18)	$x : (\lambda L : *. L \rightarrow L) B$	
	$x : (\lambda L : *. L \rightarrow L) B$	(var) on (16)
(19)	$\beta \rightarrow \beta : *$	(form) on (15) and (15)
(20)	$x : \beta \rightarrow \beta$	(conv) on (18) and (19)

Shortened derivation: we silently execute the second premise of the (conv) rule

(18)	$x : (\lambda L : *. L \rightarrow L) B$	
	$x : (\lambda L : *. L \rightarrow L) B$	(var) on (16)
(20)	$x : \beta \rightarrow \beta$	(conv) on (18)

Subject reduction, type reduction, conversion

$$\frac{\Gamma \vdash A : B \quad \downarrow_B \quad \text{subject reduction} \quad (\text{Axiom})}{\Pi \vdash A' : B} \quad A' \quad \text{Axiom}$$

$$\frac{\Gamma \vdash A : B \quad \downarrow_B \quad \text{type reduction} \quad (\text{converse of (conv)})}{\Gamma \vdash A : B'} \quad B' \quad \text{not derivable without (conv) rule}$$

$$\frac{\Gamma \vdash A : B \quad \stackrel{=B}{=} \quad B' \quad \text{conversion (conv) rule}}{\Gamma \vdash A : B', \text{ if } \Gamma \vdash B' : S}$$

## Properties of $\lambda_W$

- $\lambda_W$  satisfies all theorems that  $\lambda_L$  does, with a small modification to .
- Uniqueness of types lemma: if  $\Gamma \vdash A : B$  and  $\Gamma \vdash A : B'$ , then  $B =_B B'$  (instead of  $B \equiv B'$ )