SHIFT RADIX SYSTEMS WITH GENERAL PARAMETERS

ATTILA PETHŐ, JÖRG THUSWALDNER, AND MARIO WEITZER

Abstract. There are two dimensional expanding SRS, which have some periodic orbits. The aim of the present note is to describe as good as possible such unusual points. We give all regions, to which points belong obvious cycles, like $(1), (-1), (1,-1), (1,0), (-1,0)$. We prove that if $\mathbf{r} = (r_0, r_1) \in \mathbb{R}^2$ neither belongs to these nor to the finite region $1 \le r_0 \le 4/3, -r_0 \le r_1 < r_0 - 1$ then $\tau_{\mathbf{r}}$ has only the trivial bounded orbit $\mathbf{0}$. The further reduction should be quite involved because for all $1 \le r_0 < 4/3$ there exist at least one interval I such that for the point (r_0, r_1) this is not true, whenever $r_1 \in I$.

1. Introduction

The aim of this paper is to study properties of orbits of so-called *shift radix systems.* These objects were introduced in 2005 by Akiyama *et al.* [1]. We start by recalling their exact definition (for $x \in \mathbb{R}$ we use the notation $\lfloor x \rfloor$ and $\{x\}$ for its integer and fractional part, respectively).

Definition 1.1. For $d \in \mathbb{N}$ and $\mathbf{r} \in \mathbb{R}^d$ we call the mapping

$$\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d,$$
$$\mathbf{a} = (a_0, \dots, a_{d-1})^t \mapsto (a_1, \dots, a_{d-1}, -\lfloor \mathbf{ra} \rfloor)^t$$

the *d-dimensional shift radix system (SRS) associated with* $\mathbf{r}$.

It is easy to see from this definition that $\tau_{\mathbf{r}}$ is almost linear in the sense that it can be written as

$$\tau_{\mathbf{r}}(\mathbf{a}) = R(\mathbf{r})\mathbf{a} + (0, \dots, 0, \{\mathbf{ra}\})^t, \quad \text{where} \quad R(\mathbf{r}) = \begin{pmatrix} \mathbf{0} & & I_{d-1} & \\ -r_0 & -r_1 & \cdots & -r_d \end{pmatrix}$$

with I_{d-1} being the $(d-1) \times (d-1)$ identity matrix. However, the small deviation from linearity entails a rich dynamical behaviour of $\tau_{\mathbf{r}}$ that has already been studied extensively in the literature. For a survey on different aspects of shift radix systems we refer to [9].

For $\mathbf{a} \in \mathbb{Z}^d$ the *orbit of* $\mathbf{a}$ *under* $\tau_{\mathbf{r}}$ is given by the sequence $(\tau_{\mathbf{r}}^n(\mathbf{a}))$, where $\tau_{\mathbf{r}}^n(\mathbf{a})$ stands for the n-fold application of $\tau_{\mathbf{r}}$ to $\mathbf{a}$. By the definition of $\tau_{\mathbf{r}}$ the last $d-1$ entries of $\tau_{\mathbf{r}}^n(\mathbf{a})$ and the first $d-1$ entries of $\tau_{\mathbf{r}}^{n+1}(\mathbf{a})$ coincide for all $n \in \mathbb{N}$, hence we may choose to drop the redundant information and identify the orbit of $\mathbf{a}$ with the sequence of integers consisting of the entries of $\mathbf{a}$ followed by the last entries of $\tau_{\mathbf{r}}^n(\mathbf{a})$ only, *i.e.*, if $\mathbf{a} = (a_0, \dots, a_{d-1})^t$ and a_{d-1+n} is the last entry of $\tau_{\mathbf{r}}^n(\mathbf{a})$ for $n \in \mathbb{N}$, we identify the orbit of $\mathbf{a}$ with the sequence $(a_n)_{n \in \mathbb{N}}$. If (a_n) ultimately consists only of zeroes, we call it a *trivial* orbit of $\tau_{\mathbf{r}}$, otherwise, an orbit of $\tau_{\mathbf{r}}$ is called *nontrivial.* An orbit (a_n) is *ultimately periodic* if there exist $n_0, p \in \mathbb{N}$, $p > 0$, such that $a_{n+p} = a_n$ for $n \ge n_0$ and *periodic* if this holds for $n_0 = 0$. In this case we call $(a_n, \dots, a_{n+p-1})$ with $n \ge n_0$ a *cycle* of $\tau_{\mathbf{r}}$. The cycle (0) is called *trivial*, all other cycles are *nontrivial.* The integer p is called the *period* of the cycle.

The properties of the orbits of $\tau_{\mathbf{r}}$ were studied extensively in the literature, see *e.g.* [4, 7, 12]. In particular, define the sets

$$\mathcal{D}_d := \{\mathbf{r} \in \mathbb{R}^d \ : \text{ each orbit of } \tau_{\mathbf{r}} \text{ ends up in a cycle}\},$$
$$\mathcal{D}_d^{(0)} := \{\mathbf{r} \in \mathbb{R}^d \ : \text{ each orbit of } \tau_{\mathbf{r}} \text{ ends up in the trivial cycle}\}.$$

Date: September 18, 2017.

Research of the first author is supported in part by the OTKA grants NK104208, NK115479 and the FWF grant P27050. He also thanks the hospitality of the Montanuniversität Leoben and the Technische Universität Graz. The second author is supported by grant P29910 "Dynamics, Geometry, and Arithmetic of Numeration" of the Austrian Science Fund (FWF).

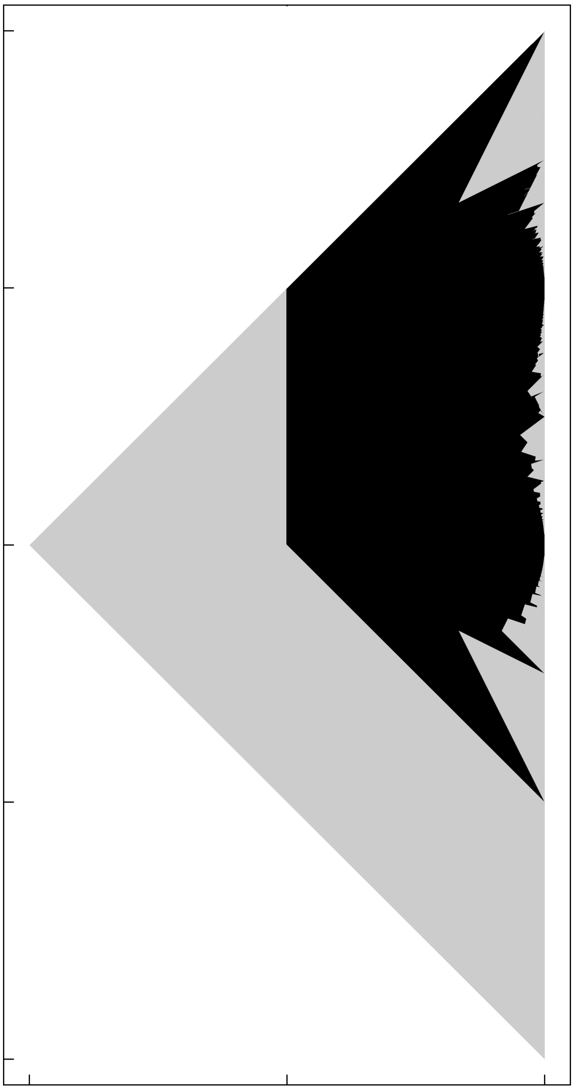

FIGURE 1. $\mathcal{D}_2^{(0)}$ (black) in $\mathcal{D}_2$ (gray).

As was already observed in [1], for each $d \in \mathbb{N}$ both of these sets are contained in the closure of the so-called Schur-Cohn region (see [11])

$$\mathcal{E}_d := \{(r_0, \ldots, r_{d-1})^t \in \mathbb{R}^d \ : \ \text{each root } y \text{ of } x^d + r_{d-1}x^{d-1} + \cdots + r_0 \text{ satisfies } |y| < 1\}.$$

In other words, the regions $\mathcal{D}_d$ and $\mathcal{D}_d^{(0)}$ only concern parameters corresponding to *contractive* polynomials. In all these cases the linear part $R(\mathbf{r})$ of $\tau_\mathbf{r}$ is contractive. Interesting results were proved also in the indifferent case, *i.e.*, if all roots of the characteristic polynomial of $R(\mathbf{r})$ are on the unit circle, see [2, 3, 6, 7, 8]. Indeed, this is the difficult part for the description of the sets $\mathcal{D}_d$.

In this paper we focus our attention to the case for which $\tau_\mathbf{r}$ is expanding. Our principal question is *for which values of* $\mathbf{r}$ *the only cycle of* $\tau_\mathbf{r}$ *is the trivial one?* In other words, we are going to study the set

$$\mathcal{D}_d^{(*)} := \{\mathbf{r} \in \mathbb{R}^d \ : \ \text{each ultimately periodic orbit of } \tau_\mathbf{r} \text{ ends up in the trivial cycle}\}.$$

Indeed, it is clear that $\mathcal{D}_d^{(0)} \subset \mathcal{D}_d^{(*)}$. However, the reverse inclusion is not true as the orbits of $\tau_\mathbf{r}$ with $\mathbf{r} \in \mathcal{D}_d^{(*)}$ may also be unbounded. We present examples later.

Our aim is to describe the sets $\mathcal{D}_d^{(*)}$. We are dealing only with the case $d = 2$. Our investigations show that the complete description of $\mathcal{D}_d^{(*)}$ is very hard (and even seems to be beyond reach) already for $d = 2$.

If we define the sequence $(e_n) \in [0,1)^{\mathbb{N}}$ by $e_n := \{\mathbf{r}\tau_{\mathbf{r}}^n(\mathbf{a})\}$, where $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of $x \in \mathbb{R}$, then it follows from the definitions that

$$a_{n+d} + r_{d-1}a_{n+d-1} + \cdots + r_0 a_n = e_n \in [0,1) \tag{1.1}$$

for all $n \in \mathbb{N}$, *i.e.*, (a_n) is a nearly linear recursive sequence, in the sequel *nlrs*. In this paper we begin with the real vector $\mathbf{r} = (r_0, \ldots, r_{d-1})^t$ and, using this vector, provide the rule $\tau_{\mathbf{r}}$ and the starting value $(a_0, \ldots, a_{d-1})^t$ in order to compute the sequence (a_n). We then study the properties of those sequences. Akiyama, Evertse and Pethő [5] considered nlrs from a different point of view. They called (a_n) a sequence of complex numbers *nearly linear recursive*, if there exist $p_0, \ldots, p_{d-1} \in \mathbb{C}$ such that the "error sequence" (e_n) defined by

$$a_{n+d} + p_{d-1}a_{n+d-1} + \cdots + p_0 a_n = e_n$$

is bounded. In their setting (a_n) is given! For a given nlrs (a_n), the set of polynomials $B_t x^t + B_{t-1}x^{t-1} + \cdots + B_0 \in \mathbb{C}[x]$ such that the sequence $(\sum_{i=0}^t B_i a_{n+i})$ is bounded is an ideal of the polynomial ring $\mathbb{C}[x]$, which is called the *ideal of* (a_n). As $\mathbb{C}[x]$ is a principal ideal ring, for all nlrs (a_n) there exists a unique polynomial, which generates the ideal of (a_n). This is called the *characteristic polynomial* of (a_n), and they proved that all roots of the characteristic polynomial of an nlrs have absolute value at least one. They also proved that if one of the roots of the characteristic polynomial of (a_n) lie outside the unit disc, then $(|a_n|)$ tends to grow exponentially.

If the mapping $\tau_{\mathbf{r}}$ is expanding then the polynomial $x^d + r_{d-1}x^{d-1} + \cdots + r_0$ has a root outside the unit disc. However it can happen, even in that case, when all of its roots lie outside the unit disc that a bounded sequence of integers (a_n) satisfies (1.1). This happens for example in the case $d = 2, p_1 = -1.15, p_0 = 1.1$, when both roots of $x^2 - 1.15x + 1.1$ are larger than 1, but the constant sequence (1) satisfies (1.1). It is easy to resolve this apparent contradiction: the characteristic polynomial of (1) is in the sense of [5] not $x^2 - 1.15x + 1.1$, but the constant polynomial 1.

In Section 2 we present preparatory results about bounded nlrs. SRS is a special case of nlrs, but have special features too. We collect them in Section 3 first generally, then specialized to $d = 2$. The long Section 4 includes the characterization of $\mathcal{D}_2^{(*)}$. We subdivided it into subsections. First we describe regions, which do not belong to $\mathcal{D}_2^{(*)}$, because they have obvious cycles, like $(1), (-1), (1,-1), (1,0), (-1,0)$. Next we turn to prove that large regions belong to $\mathcal{D}_2^{(*)}$. First we consider such regions for which the proof is simple. We can exclude the existence of cycles by proving that the orbits are monotonically increasing or decreasing. This happens always if the roots of $x^2+r_1x+r_0$ are real and at least one of them is positive. The hard cases are studied in Subsection 4.3 and 4.4. We are able to reduce to the uncertain region to a bounded one by using estimates for the size of elements of a cycle depending on the size of the roots of $x^2+r_1x+r_0$. Finally we reduce further the the uncertain region to $1 \le r_0 \le 4/3, -r_0 \le r_1 < r_0 - 1$ by using a Brunotte-type algorithm. The further reduction should be quite involved because for all $1 \le r_0 < 4/3$ there exist at least one interval I such that $(r_0, r_1) \notin \mathcal{D}_2^{(*)}$ whenever $r_1 \in I$.

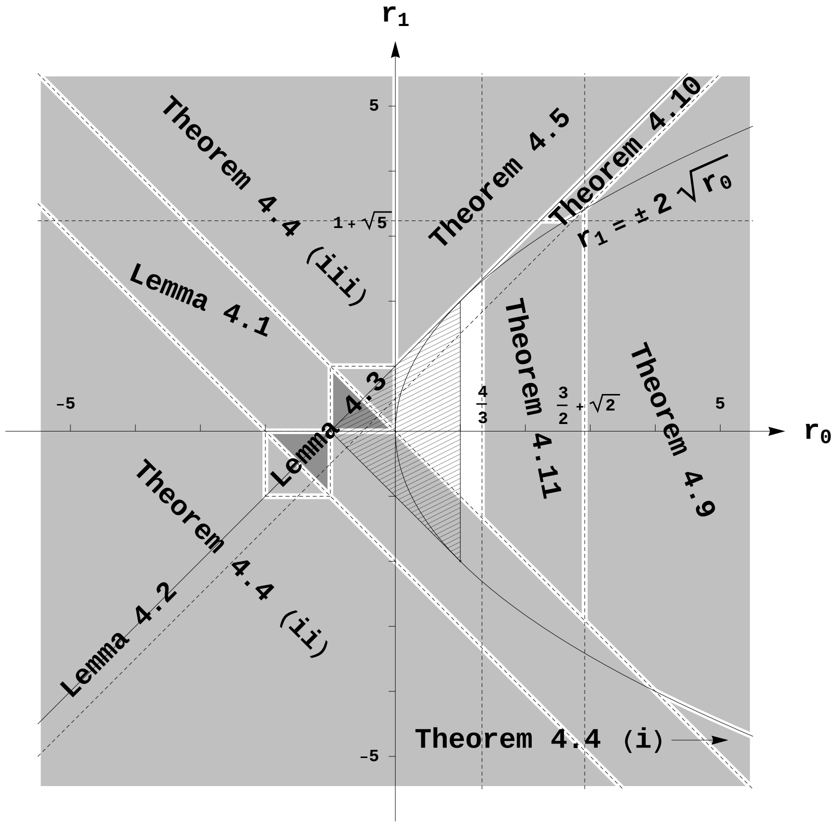

FIGURE 2. Overview

2. General results on bounded nearly linear recurrent sequences

The present section contains some preparatory results that are stated in the more general framework of nlrs. These sequences are studied thoroughly in the recent paper [5]. For the sake of completeness we recall the definition of these objects. A sequence (a_n) is called *nearly linear recurrent* if there exist $p_0, \dots, p_{d-1} \in \mathbb{C}$ such that the sequence (e_n) defined by

$$a_{n+d} + p_{d-1}a_{n+d-1} + \cdots + p_0 a_n = e_n$$

is bounded.

Our Theorem 2.2 is a kind of complement to [5, Theorem 1.1]: in the terminology of that paper it deals with nlrs with constant characteristic polynomial.

We start with a preparatory lemma.

Lemma 2.1. *Let $\beta \in \mathbb{C}$ with $|\beta| \neq 1$ and $(b_n), (e_n) \in \mathbb{C}^{\mathbb{N}}$ such that*

$$b_{n+1} - \beta b_n = e_n \tag{2.1}$$

for all $n \in \mathbb{N}$. Furthermore, assume that there is $E > 0$ such that $|e_n| \leq E$ for all $n \in \mathbb{N}$. Then the following assertions hold.

(i) *If $|\beta| < 1$, then for each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $|b_n| < \frac{E}{1-|\beta|} + \varepsilon$ holds for $n \geq n_0$. In particular, $(|b_n|)$ is bounded.*

(ii) *If $|\beta| > 1$ and $(|b_n|)$ is bounded, then there is $n_0 \in \mathbb{N}$ such that $|b_n| \leq \frac{E}{|\beta|-1}$ for $n \geq n_0$.*

Proof. Iterating (2.1) for n times yields

$$b_{n+k} - \beta^n b_k = e_{n+k-1} + \beta e_{n+k-2} + \cdots + e_k \beta^{n-1} \tag{2.2}$$

for all $n \in \mathbb{N}$ and for all $k \in \mathbb{N}$.

If $|\beta| < 1$ and $k = 1$ we thus get

$$|b_{n+1}| \le \sum_{j=1}^{n} |e_j| \, |\beta|^{n-j} + |\beta|^n \, |b_1| < \frac{E}{1-|\beta|} + |\beta|^n \, |b_1| \,. \tag{2.3}$$

Since $|\beta|^n \, |b_1|$ tends to 0 for $n \to \infty$ this proves (i).

To prove (ii) assume that $|\beta| > 1$ and that there exists $B > 0$ satisfying $|b_n| \le B$ for all $n \in \mathbb{N}$. Dividing (2.2) by β^n yields

$$|b_k| = \left| -\frac{1}{\beta} \sum_{j=0}^{n-1} e_{k+j} \beta^{-j} + \frac{b_{n+k}}{\beta^n} \right| < \frac{E}{|\beta|} \sum_{j=0}^{\infty} \frac{1}{|\beta|^j} + \frac{|b_{n+k}|}{|\beta|^n} \le \frac{E}{|\beta|-1} + \frac{B}{|\beta|^n},$$

which proves (ii) because k is fixed, and $B/|\beta|^n$ tends to 0 for $n \to \infty$. □

The following result, which is of interest in its own right contains bounds for certain nlrs.

Theorem 2.2. *Let $d \in \mathbb{N}$ and $\beta_1, \dots, \beta_d \in \mathbb{C}$ such that $|\beta_1| \le \cdots \le |\beta_r| < 1 < |\beta_{r+1}| \le \cdots \le |\beta_d|$ for some $r \in \{1, \dots, d\}$. Furthermore, let $(a_n), (e_n) \in \mathbb{C}^{\mathbb{N}}$ with $|e_n| \le E$ for all $n \in \mathbb{N}$ and some $E > 0$, such that*

$$a_{n+d} + p_{d-1} a_{n+d-1} + \cdots + p_0 a_n = e_n \tag{2.4}$$

for all $n \in \mathbb{N}$, where $(x-\beta_1)\cdots(x-\beta_d) = x^d + p_{d-1}x^{d-1} + \cdots + p_1 x + p_0$. Then,

(i) *If $r = d$, then for each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that*

$$|a_n| < \frac{E}{\prod_{j=1}^{d}(1-|\beta_j|)} + \varepsilon$$

for $n \ge n_0$. In particular, $(|a_n|)$ is bounded.

(ii) *If $r < d$ and $(|a_n|)$ is bounded, then for each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that*

$$|a_n| < \frac{E}{\prod_{j=1}^{d}|1-|\beta_j||} + \varepsilon$$

for $n \ge n_0$.

Proof. The assertion is true for $d = 1$ by Lemma 2.1. Assume that it is also true for $d-1$ and let

$$S_j(x_1, \dots, x_k) := \sum_{1 \le i_1 < \cdots < i_j \le k} x_{i_1} \cdots x_{i_j} \tag{2.5}$$

for all $k \in \mathbb{N}$ and $j \in \{1, \dots, k\}$, the j-th elementary symmetric polynomial on the indeterminates $x_1, \dots, x_k$. Furthermore, we set $S_0(x_1, \dots, x_k) := 1$ for all $k \in \mathbb{N}$ and $S_j(x_1, \dots, x_k) := 0$ for all $k \in \mathbb{N}$ and for all $j \in \mathbb{Z} \setminus \{0, \dots, k\}$. It is well known that

$$p_{d-j} = (-1)^j S_j(\beta_1, \dots, \beta_d) \tag{2.6}$$

for all $j \in \{1, \dots, d\}$. Clearly,

$$S_j(x_1, \dots, x_k) = x_k S_{j-1}(x_1, \dots, x_{k-1}) + S_j(x_1, \dots, x_{k-1}) \tag{2.7}$$

which, together with (2.4), yields

$$\begin{aligned} e_n &= \sum_{j=0}^{d} p_{d-j} a_{n+d-j} \\ &= \sum_{j=0}^{d} (-1)^j S_j(\beta_1,\dots,\beta_d) a_{n+d-j} \\ &= \sum_{j=0}^{d} (-1)^j (\beta_d S_{j-1}(\beta_1,\dots,\beta_{d-1}) + S_j(\beta_1,\dots,\beta_{d-1})) a_{n+d-j} \\ &= \sum_{j=0}^{d-1} (-1)^j S_j(\beta_1,\dots,\beta_{d-1})(a_{n+d-j} - \beta_d a_{n+d-j-1}) \\ &= \sum_{j=0}^{d-1} (-1)^j S_j(\beta_1,\dots,\beta_{d-1}) b_{n+d-j-1}, \end{aligned}$$

where $b_n := a_{n+1} - \beta_d a_n$ for all $n \in \mathbb{N}$. Clearly, if $(|a_n|)$ is bounded, then so is $(|b_n|)$ and the assumptions of the theorem hold for both sequences. By the induction hypothesis we get that for each $\varepsilon > 0$

$$|b_n| < \frac{E}{\prod_{j=1}^{d-1} |1 - |\beta_j||} + \varepsilon \tag{2.8}$$

for all large enough $n \in \mathbb{N}$. Thus, by Lemma 2.1,

$$|a_n| < \frac{E}{\prod_{j=1}^{d} |1 - |\beta_j||} + \varepsilon \tag{2.9}$$

holds for all large enough $n \in \mathbb{N}$. □

3. Bounded orbits of expansive shift radix systems

3.1. **General SRS.**

The situation for expanding and contractive $\tau_{\mathbf{r}}$ is related, as the following consequence of Theorem 2.2 indicates.

Corollary 3.1. *Assume that the sequence of integers (a_n) satisfies (1.1) for all $n \in \mathbb{N}$. Let $x^d + r_{d-1}x^{d-1} + \cdots + r_1 x + r_0 = (x - \beta_1)\cdots(x - \beta_d)$ where $\beta_1, \dots, \beta_d \in \mathbb{C}$ such that $|\beta_1| \le \cdots \le |\beta_r| < 1 < |\beta_{r+1}| \le \cdots \le |\beta_d|$ for some $r \in \{0, \dots, d\}$. Then,*

(i) *if $r = d$, or*
(ii) *if $r < d$ and $(|a_n|)$ is bounded,*

then it is ultimately periodic and

$$|a_n| \le \frac{1}{\prod_{j=1}^{d} |1 - |\beta_j||}$$

holds for all elements of the cycle.

Proof. If $r = d$ then Theorem 2.2 (i) implies that (a_n) is bounded. In the case $r < d$ the boundedness of (a_n) is part of the assumptions. Since (a_n) is a bounded sequence of integers there exist $j < k$ such that $a_{j+i} = a_{k+i}$ holds for each $i \in \{0, \dots, d-1\}$. However, by (1.1) a_{j+d} is defined uniquely in terms of $a_j, \dots, a_{j+d-1}$ and a_{k+d} is defined uniquely in terms of $a_k, \dots, a_{k+d-1}$. Thus $a_{j+d} = a_{k+d}$ and, by induction, $a_{j+n} = a_{k+n}$ for all $n \ge 0$. Thus our sequence is ultimately periodic.

Let a_j be an element of the cycle, and choose $\varepsilon > 0$ arbitrary. Then according to Theorem 2.2 there exists $n_0 \in \mathbb{N}$ such that

$$|a_n| < \frac{1}{\prod_{j=1}^{d} |1 - |\beta_j||} + \varepsilon \tag{3.1}$$

for $n \geq n_0$. Since a_j is an element of the cycle of (a_n), there is an index $n \geq n_0$ such that $a_j = a_n$. Hence, the estimate in (3.1) also holds for a_j. But since ε was arbitrary we even have

$$|a_j| \leq \frac{1}{\prod_{j=1}^{d} |1 - |\beta_j||}$$

and the proof is finished. □

3.2. **Specialization to two dimensional SRS.**
For the remaining part of the section let $d = 2$, $\mathbf{r} = (r_0, r_1)^t \in \mathbb{R}^2$, $\mathbf{a} = (a_0, a_1)^t \in \mathbb{Z}^2$, (a_n) the orbit of $\mathbf{a}$ under $\tau_{\mathbf{r}}$, and let (e_n) be the corresponding error sequence, *i.e.*,

$$a_{n+2} + r_1 a_{n+1} + r_0 a_n = e_n, \ e_n \in [0, 1) \tag{3.2}$$

for all $n \in \mathbb{N}$. Furthermore, define $\alpha_1, \alpha_2 \in \mathbb{C}$ by

$$x^2 + r_1 x + r_0 = (x - \alpha_1)(x - \alpha_2). \tag{3.3}$$

Then, $r_0 = \alpha_1\alpha_2$ and $r_1 = -(\alpha_1 + \alpha_2)$.

Proposition 3.2. *Let $|\alpha_1|, |\alpha_2| \neq 1$ and assume that $(|a_n|)$ is bounded and, hence, ultimately periodic. Then all elements a_n contained in the cycle satisfy*

$$|a_{n+1} - \alpha_i a_n| \leq \frac{1}{||\alpha_{3-i}| - 1|} \tag{3.4}$$

for $i = 1, 2$ and

$$|a_n| \leq \frac{1}{||\alpha_1| - 1| \, ||\alpha_2| - 1|}. \tag{3.5}$$

Proof. We have

$$\begin{aligned} e_n &= a_{n+2} - (\alpha_1 + \alpha_2)a_{n+1} + \alpha_1\alpha_2 a_n \\ &= (a_{n+2} - \alpha_1 a_{n+1}) - \alpha_2(a_{n+1} - \alpha_1 a_n) = b_{n+1} - \alpha_2 b_n, \end{aligned} \tag{3.6}$$

with $b_n := a_{n+1} - \alpha_1 a_n, n \geq 0$. By Corollary 3.1 (a_n) is ultimately periodic, hence, (b_n) is ultimately periodic as well. With these sequences (b_n) and (e_n), and with the choices $\beta = \alpha_2$ and $E = 1$ the assumptions of Lemma 2.1 are satisfied, thus for each $\varepsilon > 0$ we have

$$|a_{n+1} - \alpha_1 a_n| = |b_n| < \frac{1}{|1 - |\alpha_2||} + \varepsilon$$

for n sufficiently large. For b_n in the cycle we can get rid of the summand ε by the same reasoning as in the proof of Corollary 3.1 and (3.4) follows for the case $i = 1$. Interchanging α_1 and α_2 we get the case $i = 2$.

Inequality (3.5) is a special instance of Corollary 3.1. □

The following result is an immediate consequence of Proposition 3.2.

Corollary 3.3. *If $\frac{1}{||\alpha_1|-1|} \frac{1}{||\alpha_2|-1|} < 1$ then each orbit of $\tau_{\mathbf{r}}$ is either unbounded or ends up in the cycle* (0).

Proposition 3.2 holds for arbitrary complex numbers α_1, α_2. If they are real then we can improve inequality (3.4) a bit, which has advantage in the applications.

Proposition 3.4. *Let $\alpha_1, \alpha_2 \neq \pm 1$ be real numbers. Assume that $(|a_n|)$ is bounded and, hence, ultimately periodic. Then all elements a_n contained in the cycle satisfy*

$$0 \leq a_{n+1} - \alpha_1 a_n < \frac{1}{1 - \alpha_2}, \ \textit{if} \quad 0 \leq \alpha_2 < 1, \tag{3.7}$$

$$0 \leq -(a_{n+1} - \alpha_1 a_n) < \frac{1}{\alpha_2 - 1}, \ \textit{if} \quad \alpha_2 > 1, \tag{3.8}$$

$$\frac{\alpha_2}{1 - \alpha_2^2} < a_{n+1} - \alpha_1 a_n < \frac{1}{1 - \alpha_2^2}, \ \textit{if} \quad -1 < \alpha_2 < 0, \tag{3.9}$$

$$\frac{\alpha_2}{\alpha_2^2 - 1} < -(a_{n+1} - \alpha_1 a_n) < \frac{1}{\alpha_2^2 - 1}, \ \textit{if} \quad \alpha_2 < -1. \tag{3.10}$$

Proof. We may assume w.l.o.g that (a_n) is purely periodic with period length p. Set, as in the proof of Proposition 3.2 $b_n = a_{n+1} - \alpha_1 a_n$ for $n \ge 0$. Then (b_n) is periodic too with period length p. Consider the equations

$$b_{j+1} - \alpha_2 b_j = e_j,$$

(c.f. (3.6)) for $j = 0, \dots, p-1$ and notice $b_p = b_0$. Multiplying the equations by appropriate powers of α_2 we get

$$\alpha_2^j b_{p-j} - \alpha_2^{j+1} b_{p-j-1} = \alpha_2^j e_{p-j-1},\ j = 0, \dots, p-1. \tag{3.11}$$

If $\alpha_2 > 0$ then summing these equations and taking $0 \le e_j < 1$ into account we get

$$0 \le b_0(1 - \alpha_2^p) < \sum_{j=0}^{p-1} \alpha_2^j = \frac{1 - \alpha_2^p}{1 - \alpha_2}.$$

Distinguishing the cases $0 < \alpha_2 < 1$ and $\alpha_2 > 1$ we get the first two inequalities.

If $\alpha_2 < 0$ then we assume for simplicity p even, say $p = 2p_1$. This is allowed, because if p is a period length of a sequence then $2p$ ia a period length too. Equations (3.11) imply

$$0 \le \alpha_2^j b_{p-j} - \alpha_2^{j+1} b_{p-j-1} < \alpha_2^j,$$

if j is even, and

$$\alpha_2^j < \alpha_2^j b_{p-j} - \alpha_2^{j+1} b_{p-j-1} \le 0,$$

if j is odd. Summing these inequalities for $j = 0, \dots, 2p_1 - 1$ we obtain

$$\alpha_2 \sum_{j=0}^{p_1-1} \alpha_2^{2j} < b_0(1 - \alpha_2^p) < \sum_{j=0}^{p_1-1} \alpha_2^{2j}.$$

Using

$$\sum_{j=0}^{p_1-1} \alpha_2^{2j} = \frac{1 - \alpha_2^p}{1 - \alpha_2^2}$$

and distinguishing the cases $-1 < \alpha_2 < 0$ and $\alpha_2 < -1$ we get the last two inequalities. □

4. Characterization of $\mathcal{D}_2^{(*)}$

4.1. Regions outside of $\mathcal{D}_2^{(*)}$.

Lemma 4.1. *The mapping $\tau_{\mathbf{r}}$ has a nontrivial cycle whose elements all have the same sign if and only if $-2 < r_0 + r_1 < 0$. Thus the set $\{(r_0, r_1) \in \mathbb{R}^2 : -2 < r_0 + r_1 < 0\}$ has empty intersection with $\mathcal{D}_2^{(*)}$*

In particular, if $-1 \le r_0 + r_1 < 0$ then (1) is a cycle of $\tau_{\mathbf{r}}$, and if $-2 < r_0 + r_1 < -1$ then (-1) is a cycle of $\tau_{\mathbf{r}}$.

Proof. Assume that $\tau_{\mathbf{r}}$ has a nontrivial cycle $(a_0, \dots, a_{p-1})$ whose members have the same sign. Then,

$$0 \le a_i r_0 + a_{i+1} r_1 + a_{i+2} < 1 \tag{4.1}$$

for all $i \in \{0, \dots, p-1\}$. Summing up these inequalities and taking into account that $a_p = a_0$ and $a_{p+1} = a_1$ we get

$$0 \le \sum_{i=0}^{p-1} a_i (r_0 + r_1 + 1) < p. \tag{4.2}$$

Since all a_i have the same sign (and cannot be 0 by nontriviality of the cycle) it follows that

$$\left| \sum_{i=0}^{p-1} a_i \right| \ge p \tag{4.3}$$

and, hence,

(4.4) $$-1 < r_0 + r_1 + 1 < 1.$$

If $-1 \le r_0 + r_1 < 0$, *i.e.*, $0 \le r_0 + r_1 + 1 < \frac{1}{h}$ for $h \in \mathbb{N}$, $h > 0$, then (t) where $0 < t \le h$ is a cycle of $\tau_{\mathbf{r}}$. In particular, (1) is a cycle of $\tau_{\mathbf{r}}$ if $-1 \le r_0 + r_1 < 0$.

If $-2 < r_0 + r_1 < -1$, i.e. $-\frac{1}{h} < r_0 + r_1 + 1 < 0$ for $h \in \mathbb{N}$, $h > 0$, then $0 < tr_0 + tr_1 + t < -\frac{t}{h} \le 1$ for $-h \le t < 0$, hence (t) is an cycle of $\tau_{\mathbf{r}}$. In particular, (-1) is a cycle of $\tau_{\mathbf{r}}$ if $-1 \le r_0 + r_1 + 1 < 0$. □

Lemma 4.2. *If $\tau_{\mathbf{r}}$ has a cycle of alternating signs, then $|r_0 - r_1 + 1| < \frac{1}{2}$. Furthermore, $\tau_{\mathbf{r}}$ has a cycle of the form $(t, -t)$ for some $t \in \mathbb{Z}\setminus\{0\}$ if and only if $r_0 - r_1 + 1 = 0$. Thus $\{(r_0, r_0+1)^t : r_0 \in \mathbb{R}\}$ has empty intersection with $\mathcal{D}_2^{(*)}$.*

Proof. Assume that $\tau_{\mathbf{r}}$ has a cycle $(a_0, \dots, a_{p-1})$ with $a_i a_{i+1} < 0$ for all $i \in \{0, \dots, p-1\}$. Then the period p has to be even and we may assume without loss of generality that $a_0 > 0$. By definition,

(4.5) $$0 \le a_i r_0 + a_{i+1} r_1 + a_{i+2} < 1$$

for all $i \in \{0, \dots, p-1\}$. Multiplying the inequalities corresponding to odd values of i by -1 and summing over the whole cycle we get

(4.6) $$-\frac{p}{2} < \sum_{i=0}^{p-1} (-1)^i a_i (r_0 - r_1 + 1) < \frac{p}{2}.$$

Observing that $\sum_{i=0}^{p-1} (-1)^i a_i \ge p$ for all cycles $(a_0, \dots, a_{p-1})$ with members of alternating signs we obtain the first statement.

Assume that $\tau_{\mathbf{r}}$ admits a cycle $(t, -t)$ with $t \in \mathbb{Z} \setminus \{0\}$. We may assume $t > 0$. The inequalities

$$0 \le tr_0 - tr_1 + t < 0 \text{ and } 0 \le -tr_0 + tr_1 - t < 0$$

imply $r_0 - r_1 + 1 = 0$, as stated. □

Lemma 4.3.

- $\mathbf{r} \in \{(r_0, r_1)^t \in \mathbb{R}^2 : -2 < r_0 \le -1, -1 < r_1 \le 0\}$ *implies that $(0, -1)$ is a cycle of $\tau_{\mathbf{r}}$.*
- $\mathbf{r} \in \{(r_0, r_1)^t \in \mathbb{R}^2 : -1 \le r_0 < 0, 0 \le r_1 < 1\}$ *implies that $(0, 1)$ is a cycle of $\tau_{\mathbf{r}}$.*

Thus both of these sets have empty intersection with $\mathcal{D}_2^{(*)}$

Proof. Simple computation. □

4.2. Subregions of $\mathcal{D}_2^{(*)}$: simple cases.

Theorem 4.4. *Assume that $\mathbf{r} = (r_0, r_1)^t$ is contained in one of the following sets.*

(i) $\{(r_0, r_1)^t \in \mathbb{R}^2 : r_0 > 0, r_1 \le -2\sqrt{r_0}, r_0 + r_1 \ge 0\}$.
(ii) $\{(r_0, r_1)^t \in \mathbb{R}^2 : r_0 + r_1 \le -2, r_0 - r_1 \ne -1\} \setminus \{(r_0, r_1)^t \in \mathbb{R}^2 : r_0 > -2, r_1 > -1\}$.
(iii) $\{(r_0, r_1)^t \in \mathbb{R}^2 : r_0 + r_1 \ge 0, r_0 < 0, r_1 \ge 1\}$.

Then $\mathbf{r} \in \mathcal{D}_2^{(*)}$.

Proof. For the entire proof assume without loss of generality that $|\alpha_1| \le |\alpha_2|$. Notice that in all cases α_1 and α_2 are real.

The proof will be done by contradiction. Thus we assume that there is a non-trivial cycle $(a_0, \dots, a_{p-1})$ of $\tau_{\mathbf{r}}$.

Proof of (i): The assumptions on r_0 and r_1 in (i) imply that $1 < \alpha_1 \le \alpha_2$ (see Figure 2). Inequality (3.8) tells us $a_{n+1} \le \alpha_1 a_n$ for all $n \in \mathbb{N}$. If $a_j \le 0$ for some $j \in \{0, \dots, p-1\}$ then (a_n) is a strictly decreasing sequence of negative numbers which is impossible. Thus $a_j > 0$ for all $j \in \{0, \dots, p-1\}$. However, in this case we have $r_0 + r_1 < 0$ by Lemma 4.1 which contradicts the assumption.

Proof of (ii): If $r_0 = 0$ then $0 \le r_1 a_n + a_{n+1} < 1$ with $r_1 \le -2$ and the result follows immediately. We divide up the remaining region into four subregions.

Case (iia): Assume $r_0+r_1 \le -2$ and $r_0 > 0$. In this case we have $0 < \alpha_1 < 1 < \alpha_2$. As in (i) we can conclude that $a_{n+1} \le \alpha_1 a_n$ for all $n \in \mathbb{N}$. If $a_0 \ge 0$ then (a_n) is a strictly decreasing sequence which is impossible, hence $a_0 < 0$. But then all a_n are negative and we can apply Lemma 4.1 again to get a contradiction to our assumption $r_0 + r_1 \le -2$.

Case (iib): Assume $r_0 + r_1 \le -2$, and $r_0 < 0$, $r_1 \le -1$, and $r_0 - r_1 + 1 > 0$. Here we have $-1 < \alpha_1 < 0$ and $\alpha_2 > 1$. By Lemma 4.1 the period $(a_0, \dots, a_{p-1})$ has to have both negative and non-negative members. By $\alpha_2 > 1$ we have $a_{n+1} \le \alpha_1 a_n$ for all $n \in \mathbb{N}$ as in (i). Thus, if $a_n \ge 0$ then $a_{n+1} < 0$.

Now we exclude the possibility of $a_n = 0$ for some n. Supposing the contrary we may assume w.l.o.g. $a_0 = 0$. Then $a_1 \le \alpha_1 a_0 = 0$, but $a_1 = 0$ is excluded because otherwise (a_n) would be the zero sequence. Thus $a_1 < 0$, and we get $a_2 < -r_1 a_1 + 1 \le a_1 + 1 \le 0$ by (3.2). Repeated application of (3.2) shows that all members of the period are negative which is impossible.

Thus $a_n \ne 0$ for all $n \in \mathbb{N}$ which means that consecutive members of the period have different signs. Hence p is even and we may assume w.l.o.g that $0 < a_0 \le a_{2l}$ for all $l \in \left\{0, \dots, \frac{p}{2} - 1\right\}$. Then we have

$$a_0 r_0 + a_1 r_1 + a_2 = a_0(r_0 - r_1 + 1) + (a_0 + a_1) r_1 + (a_2 - a_0) \tag{4.7}$$

and since $r_0 - r_1 + 1 > 0$ and $a_2 \ge a_0$ the first and the third summands must be non-negative. Furthermore, we have $a_2 \le \alpha_1 a_1 < -a_1$ which implies $a_0 + a_1 \le a_2 + a_1 < 0$. Altogether we get

$$a_0 r_0 + a_1 r_1 + a_2 \ge -(a_0 + a_1) \ge 1 \tag{4.8}$$

which is a contradiction.

Case (iic): Assume that $r_0 \le -2$, $r_1 \le 0$ and $r_0 - r_1 < -1$. This implies $\alpha_1 < -1$ and $\alpha_2 > 1$. As before the period $(a_0, \dots, a_{p-1})$ has to have both negative and non-negative members by Lemma 4.1 and we also have $a_{n+1} \le \alpha_1 a_n$ for all $n \in \mathbb{N}$. Thus the in absolute value largest element of the period must be negative. Let $A := \max\{|a_j| \mid j \in \{0, \dots, p-1\}\}$ and assume w.l.o.g that $|a_0| = A$. Equation (4.7) holds again. We have $a_0(r_0 - r_1 + 1) > 0$ and $a_2 - a_0 \ge 0$. Furthermore, $a_0 + a_1 \ge 0$ implies $a_1 \ge -a_0 = |a_0| = A$ which is absurd, hence $a_0 + a_1 < 0$. Thus all summands of (4.7) are non-negative.

If either $a_2 \ne a_0$ or $r_1 \le -1$ we get $a_0 r_0 + a_1 r_1 + a_2 \ge 1$ which is a contradiction. Thus $a_0 = a_2$ and $-1 < r_1 \le 0$, Hence

$$a_0 r_0 + a_1 r_1 + a_2 = a_0(r_0 + 1) + a_1 r_1 \ge -a_0 + a_1 r_1 \ge -a_0 - |a_1| \tag{4.9}$$

where we used first $r_0 \le -2$, and then $|r_1| < 1$. The RHS of the last inequality is at least one, which is absurd, except when $-a_0 - |a_1| = 0$ or $-a_0 = |a_1|$. The case $a_1 > 0$ is impossible because $a_0 = a_2 \le \alpha_1 a_1 < -a_1$. Hence $a_1 < 0$, but then $a_1 = a_0 = a_2$ and the cycle is constant contradicting Lemma 4.1.

Case (iid): Assume that $r_0 + r_1 \le -2$ and $r_1 > 0$. Then $\alpha_1 > 1$, $\alpha_2 < -1$, and $\alpha_1 < -\alpha_2$. Interchanging the roles of α_1 and α_2 we obtain $a_{n+1} \le \alpha_2 a_n$, as in (i).

Setting $A = \max\{|a_j|, j = 0, \dots, p-1\}$ we can show as in case (iic) that if $|a_j| = A$, then $a_j < 0$. We may assume w.l.o.g $|a_0| = A$. Set

$$S = a_0 r_0 + a_1 r_1 + a_2 = a_0(r_0 + r_1 + 1) + r_1(a_1 - a_0) + a_2 - a_0.$$

We have $r_0 + r_1 + 1 \le -1$, thus the first summand is at least $-a_0 \ge 1$. As $|a_0| = -a_0 \ge |a_j|$ for all $j \ge 1$ we have $a_1 - a_0, a_2 - a_0 \ge 0$. Finally, as $r_1 > 0$ we conclude $S \ge 1$, which is a contradiction.

Proof of (iii): Assume $r_0+r_1 \ge 0$, and $r_0 < 0$, $r_1 > 1$. Then $0 < \alpha_1 < 1, \alpha_2 < -1$. Interchanging the roles of α_1 and α_2 inequality (3.7) implies $a_{n+1} > \alpha_2 a_n$. By Lemma 4.1 the sequence (a_n) has both non-positive and positive members. Thus if $a_{n-1} \le 0$ then $a_n > |a_{n-1}| \ge 0$. We also have

$$a_{n+1} < -r_0 a_{n-1} - r_1 a_n + 1 < -a_n + (1 - r_0 a_{n-1}).$$

As $1 - r_0 a_{n-1} < 1$, and $a_n, a_{n+1} \in \mathbb{Z}$ we get $a_{n+1} \le -a_n$. Hence $(|a_n|)$ is monoton increasing and it has a jump, when $a_n \le 0$. This is impossible with a periodic sequence. □

The proof of the next result, which characterizes a further region that is free from non-trivial cycles, is subdivided in several lemmas and will constitute the remaining part of the present section.

Theorem 4.5. *We have* $\{(r_0, r_1)^t \in \mathbb{R}^2 \,:\, r_0 - r_1 < -1,\, r_0 > 0\} \subset \mathcal{D}_2^{(*)}$.

In the remaining part of this section we assume that $r_0 - r_1 < -1$ and $r_0 > 0$. In this case we have $-1 < \alpha_1 < 0, \alpha_2 < -1$. We define the following sets

$$\begin{aligned}
S_0 &= \{(0,0)^t\},\\
S_1 &= \{(a_1,a_2)^t \,:\, a_1 \ge 0, a_2 \le -a_1\} \setminus \{(0,0)^t\}\\
S_2 &= \{(a_1,a_2)^t \,:\, a_1 \le 0, a_2 \ge -a_1\} \setminus \{(0,0)^t\}\\
S_3 &= \{(a_1,a_2)^t \,:\, a_1 > 0, a_2 \ge 0\}\\
S_4 &= \{(a_1,a_2)^t \,:\, a_1 < 0, a_2 \le 0\}\\
S_5 &= \{(a_1,a_2)^t \,:\, a_1 \ge 0, 0 < a_2 < -a_1\}\\
S_6 &= \{(a_1,a_2)^t \,:\, a_1 < 0, 0 < a_2 < -a_1\}.
\end{aligned}$$

It is easy to see that $S_0, \dots, S_6$ forms a partition of $\mathbb{Z}^2$.

Lemma 4.6. *We have* $\tau_{\mathbf{r}}(S_1) \subset S_2$ *and* $\tau_{\mathbf{r}}(S_2) \subset S_1$.

Proof. Let $(a_1,a_2)^t \in S_1$ and $\tau_{\mathbf{r}}(a_1,a_2) = (a_2,a_3)^t$. Then $a_3 = -\lfloor r_0a_1 + r_1a_2 \rfloor$. By the conditions on $(r_0,r_1)^t$ and $(a_1,a_2)^t$ we have

$$\begin{aligned}
a_3 &\ge -r_0a_1 - r_1a_2\\
&> -r_0a_1 - (r_0+1)a_2; \text{ (since } r_0+1 < r_1 \text{ and } a_2 < 0)\\
&\ge a_2r_0 - (r_0+1)a_2; \text{ (since } a_2 \le -a_1)\\
&= -a_2.
\end{aligned}$$

We proved $a_3 > -a_2$, which together with $a_2 < 0$ implies $(a_2,a_3)^t \in S_2$, hence the first inclusion is proved.

To prove the second inclusion let a_1, a_2, a_3 be such that $(a_1,a_2)^t \in S_2$, and $a_3 = -\lfloor r_0a_1 + r_1a_2 \rfloor$. Now we have

$$\begin{aligned}
a_3 &< -r_0a_1 - r_1a_2 + 1\\
&\le -r_0a_1 - (r_0+1)a_2 + 1; \text{ (since } r_0+1 < r_1 \text{ and } a_2 \ge 0)\\
&\le a_2r_0 - (r_0+1)a_2 + 1; \text{ (since } a_2 \ge -a_1)\\
&= -a_2 + 1.
\end{aligned}$$

Thus $a_3 \le -a_2$, which together with $a_2 > 0$ implies $(a_2,a_3)^t \in S_1$. □

Lemma 4.7. *We have* $\tau_{\mathbf{r}}(S_3) \subset S_1$ *and* $\tau_{\mathbf{r}}(S_4) \subset S_2 \cup S_0$.

Proof. Let $(a_1,a_2)^t \in S_3$, i.e. $a_1 > 0, a_2 \ge 0$. Then $a_3 = -\lfloor r_0a_1 + r_1a_2 \rfloor < -a_2$, which means $(a_2,a_3)^t = \tau_{\mathbf{r}}(a_1,a_2) \in S_1$, and the first assertion is proved.

Let now $(a_1,a_2)^t \in S_4$, which means $a_1 < 0, a_2 \le 0$. Put $a_3 = -\lfloor r_0a_1 + r_1a_2 \rfloor$. If $a_2 < 0$ then $a_3 > -a_2$, i.e. $(a_2,a_3)^t \in S_2$. If however $a_2 = 0$ then $a_3 \ge 0$, thus $(a_2,a_3)^t \in S_2 \cup S_0$. □

Lemma 4.8. *We have* $\tau_{\mathbf{r}}(S_5) \subset S_2 \cup S_4 \cup S_6$ *and* $\tau_{\mathbf{r}}(S_6) \subset S_1 \cup S_3 \cup S_5$.

Proof. Let $(a_1,a_2)^t \in S_5$ and $\tau_{\mathbf{r}}(a_1,a_2) = (a_2,a_3)^t$. Since $a_2 < 0$ the pair $(a_2,a_3)^t$ belongs to $S_2 \cup S_4 \cup S_6$.

If $(a_1,a_2)^t \in S_6$ then $a_2 > 0$ and $\tau_{\mathbf{r}}(a_1,a_2)$ belongs to $S_1 \cup S_3 \cup S_5$. □

Now we are in the position to prove Theorem 4.5. Lemmas 4.6, 4.7 and 4.8 show that the orbits of $\tau_{\mathbf{r}}$ are all governed by the following graph.

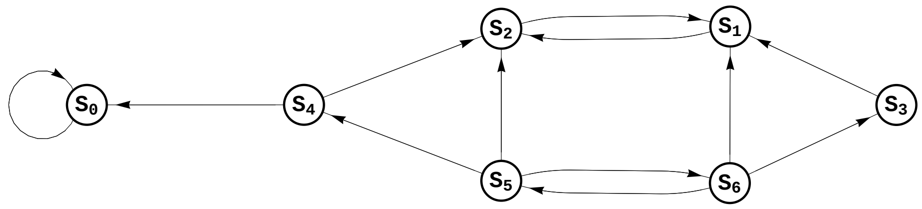

FIGURE 3. Illustration of the action of $\tau_{\mathbf{r}}$ for the proof of Theorem 4.5.

Each orbit has to end up in one of the following cycles of the graph:

(a) in $S_0 \to S_0$,
(b) in $S_1 \to S_2 \to S_1$,
(c) or in $S_5 \to S_6 \to S_5$.

In case (a) we are done.

In case (b), as soon as we arrive in the cycle $S_1 \to S_2 \to S_1$ we have according to Lemma 4.6

$$0 < a_1 \le -a_2 < a_3 \le -a_4 < \dots,$$

thus $|a_k| \to \infty$, and $\tau_{\mathbf{r}}$ has no cycle for this orbit.

In case (c) we have

$$|a_k| > |a_{k+1}| > |a_{k+2}| \dots,$$

however, since a_k is finite, this cannot go on infinitely long. Thus no orbit can end up in this cycle. □

4.3. **Subregions of $\mathcal{D}_2^{(*)}$: hard cases.**
So far only a small part of the quadrant $\{(r_0, r_1)^t \in \mathbb{R}^2 : -r_0 - 1 < r_1 < r_0 + 1\}$ was treated. The points strictly inside the triangle with vertices $(-1,0)^t, (1,-2)^t, (1,2)^t$ define contractive mappings. Thus these points are part of the classification problem: which of them belong to the set $\mathcal{D}_2^{(0)}$. The points between the parallel lines $r_1 = -r_0 - 1$ and $r_1 = -r_0$ have the finite orbit (1). Finally by Theorem 4.4 (i) the mappings corresponding to points of the region $r_0 > 0$, $r_1 \le -2\sqrt{r_0}$, and $r_0 + r_1 \ge 0$ belong to $\mathcal{D}_2^{(*)}$. In what follows we deal with the remaining part of this quadrant up to some finite region.

Approaching the critical line segment $r_0 = 1, -2 \le r_1 \le 2$ one can find points $\mathbf{r} = (r_0, r_1)^t$ such that $\tau_{\mathbf{r}}$ is expanding, but has arbitrary long cycles. Indeed Akiyama and Pethő [6] proved that the mapping $\tau_{\mathbf{r}}$ has for $\mathbf{r} = (1, r_1)^t$ for arbitrary $-2 < r_1 < 2$ infinitely many cycles. Let r_1 be irrational, and $(a_0, \dots, a_{p-1})$ be a cycle of $\tau_{\mathbf{r}}$. Then there exists $0 < \delta$ such that $\delta < a_{k-1} + r_1 a_k + a_{k+1} < 1 - \delta$ holds for all $k = 1, \dots, p$. Choosing a small enough $\varepsilon > 0$ we get $0 \le (1+\varepsilon)a_{k-1} + r_1 a_k + a_{k+1} < 1$, i.e. $(a_0, \dots, a_{p-1})$ is a non-trivial periodic orbit of $\tau_{(1+\varepsilon, r_1)}$ too.

Weitzer [12] defined six infinite sequences of polygons, which cover the critical line and each SRS associated to points in these polygons has cycles. Moreover, most of these polygons have points not only on and to the left, but also to the right of the critical line.

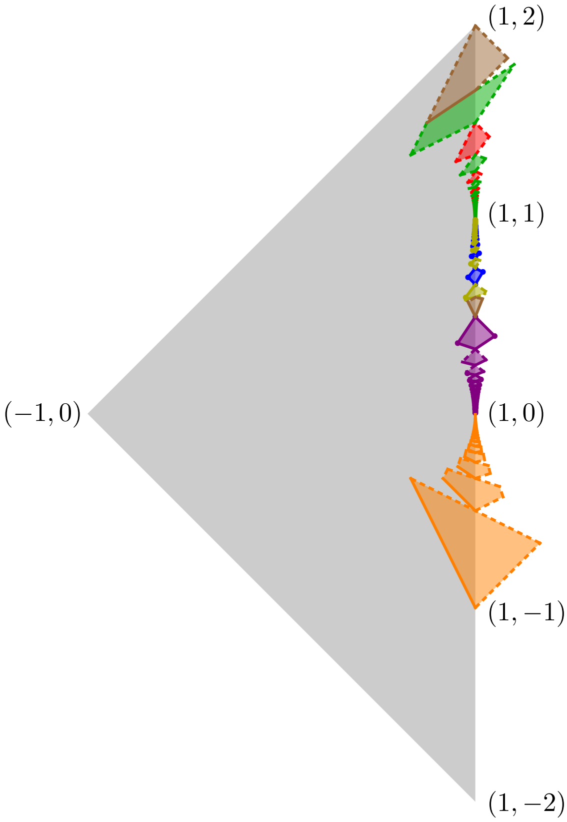

FIGURE 4. Six families of cutout polygons covering the critical line $r_0 = 1$ almost everywhere for $-1 \le r_1 \le 2$.

In what follows we prove that the points with this property belong to a bounded region.

Theorem 4.9. *We have*

$$\left\{(r_0, r_1)^t \in \mathbb{R}^2 \,:\, r_0 - r_1 > -\frac{1}{2},\, r_1 \ge \max\{-2\sqrt{r_0}, -r_0\},\, r_0 > \frac{3}{2} + \sqrt{2}\right\} \subset \mathcal{D}_2^{(*)}.$$

Proof. Let $\mathbf{r} = (r_0, r_1)^t$ be an element of the set specified in the statement of the theorem. We first deal with the case where the polynomial $P(x) = x^2 + r_1 x + r_0$ has two real roots α_1 and α_2. Then $\alpha_2 \le \alpha_1 < -1$. Assume that $\tau_{\mathbf{r}}$ admits the cycle (a_n). We have

$$|a_n| \le \frac{1}{||\alpha_1| - 1||| \alpha_2| - 1|} = \frac{1}{(\alpha_1 + 1)(\alpha_2 + 1)} = \frac{1}{r_0 - r_1 + 1} < 2$$

by Corollary 3.1. Thus $\tau_{\mathbf{r}}$ only admits cycles consisting of elements taken from the set $\{-1, 0, 1\}$. A simple computation shows that this is impossible.

Now we switch to the case where $P(x)$ has a pair of complex conjugate roots, *i.e.*, $\alpha_2 = \bar{\alpha}_1$, then using Corollary 3.1 again we obtain

$$|a_n| \le \frac{1}{||\alpha_1| - 1||| \alpha_2| - 1|} = \frac{1}{r_0 - 2\sqrt{r_0} + 1} = \frac{1}{(\sqrt{r_0} - 1)^2}$$

because $|\alpha_1| = |\alpha_2|$ and $\alpha_1 \alpha_2 = r_0$. Since $r_0 > \frac{3}{2} + \sqrt{2}$ is equivalent to $\sqrt{r_0} - 1 > \frac{1}{\sqrt{2}}$ this implies that $|a_n| < 2$ and we get the contradiction as above. (Note that the line $r_1 = r_0 + \frac{1}{2}$ intersects the parable $r_1^2 = 4r_0$ in the points $(\frac{3}{2} \pm \sqrt{2}, 2 \pm \sqrt{2})^t$.) □

With more effort one can improve the last theorem, but a complete characterization of points without non-trivial periodic points is, in spite of the results of Weitzer [12], and of Akiyama and Pethő [6] mentioned above, a very hard problem.

It remains one more infinite region, namely the points of the strip between the lines $r_0 - r_1 = -1$ and $r_0 - r_1 = -\frac{1}{2}$ over the parable $r_1^2 = 4r_0$. By our last theorem in this section only a bounded part of it may have points which associated SRS has non-trivial cycles.

Theorem 4.10. *We have*

$$\left\{(r_0, r_1)^t \in \mathbb{R}^2 \,:\, -1 < r_0 - r_1 \le -\frac{1}{2},\, r_1 \ge 1 + \sqrt{5},\, r_1^2 \ge 4r_0\right\} \subset \mathcal{D}_2^{(*)}.$$

Proof. Let $\mathbf{r} = (r_0, r_1)^t$ be an element of the set specified in the statement of the theorem and assume that (a_n) is a non-trivial periodic sequence. By the assumptions, the roots α_1, α_2 of $x^2 + r_1 x + r_0$ are real and satisfy $\alpha_2 \le \alpha_1 < -1$. As $\alpha_1 + \alpha_2 = -r_1 \le -(1 + \sqrt{5})$ we have $\alpha_2 \le -\frac{1+\sqrt{5}}{2}$. Thus we obtain

$$\frac{-1}{\alpha_2^2 - 1} < a_{n+1} - \alpha_1 a_n < \frac{-\alpha_2}{\alpha_2^2 - 1}$$

by (3.10).

The function $\frac{-x}{x^2-1}$ is monotonically increasing in $(-\infty, -\frac{1+\sqrt{5}}{2}]$, thus it takes its maximum at $x = -\frac{1+\sqrt{5}}{2}$, which is 1. This implies that $\frac{-1}{\alpha_2^2-1} \ge \frac{1}{\alpha_2} \ge -\frac{2}{1+\sqrt{5}} = \frac{1-\sqrt{5}}{2}$, thus we have

$$-1 < a_{n+1} - \alpha_1 a_n < 1.$$

This inequality implies $a_n \neq 0$ for all n and consecutive members of the sequence (a_n) must have different sign. Moreover we can write it in the form

$$(\alpha_1 + 1)a_n - 1 < a_{n+1} + a_n < (\alpha_1 + 1)a_n + 1.$$

If $a_n < 0$ then $(\alpha_1 + 1)a_n > 0$ and $a_{n+1} + a_n \ge 0$, *i.e.*, $a_{n+1} \ge -a_n$.

If $a_n > 0$ then $(\alpha_1 + 1)a_n < 0$ and $a_{n+1} + a_n \le 0$, *i.e.*, $a_{n+1} \le -a_n$. Hence the sequence $(|a_n|)$ is monotonically increasing. It can be periodic only if it is a constant sequence, *i.e.*, if $(a_n) = (a, -a, a, -a, \ldots)$ for some $a \in \mathbb{Z}$. However this is excluded by Lemma 4.2. This gives the desired contradiction. □

4.4. Subregions of $\mathcal{D}_2^{(*)}$: algorithmic approaches.

In the previous sections we characterized $\mathcal{D}_2^{(*)}$ up to a bounded region which we denote by $\mathcal{R} \subseteq \mathbb{R}^2$. $\mathcal{R} \cap \mathrm{int}(\mathcal{D}_2)$ has been characterized in large parts in [12] by two algorithms which can be adapted to also work for those parts of the exterior of $\mathcal{D}_2$ for which the corresponding SRS is expanding. Doing so yields the following result:

Theorem 4.11. *If $\mathcal{D} \subseteq \mathbb{R}^2$ denotes the region not covered by any of the theorems above, then $\mathcal{D} \cap \left\{(x, y) \in \mathbb{R}^2 \mid x \ge \frac{4}{3}\right\}$ is contained in $\mathcal{D}_2^{(*)}$.*

Proof. We outline the idea behind the adapted version of one of the algorithms from [12]. We need the algorithms to work for parameters of SRS which are expanding instead of contracting. The first ingredient is a result by Lagarias and Wang [10] which states that for any $\mathbf{r} \in \mathbb{R}^2$ for which

$$R(\mathbf{r}) = \begin{pmatrix} 0 & 1 \\ -r_0 & -r_1 \end{pmatrix}$$

is expanding and any $1 < \rho < \min\{|\lambda| \,:\, \lambda \text{ eigenvalue of } R(\mathbf{r})\}$, there is a norm $\|\cdot\|_{\mathbf{r},\rho}$ on $\mathbb{R}^2$ such that $\|R(\mathbf{r})\mathbf{x}\|_{\mathbf{r},\rho} > \rho \|\mathbf{x}\|_{\mathbf{r},\rho}$ for all $\mathbf{x} \in \mathbb{Z}^2$. If $\|\mathbf{x}\|_{\mathbf{r},\rho} > \frac{\|(0,\ldots,0,1)\|_{\mathbf{r},\rho}}{\rho-1}$ we thus get

$$\|\tau_{\mathbf{r}}(\mathbf{x})\|_{\mathbf{r},\rho} \ge \|R(\mathbf{r})\mathbf{x}\|_{\mathbf{r},\rho} - \|(0,\ldots,0,1)\|_{\mathbf{r},\rho} > \|\mathbf{x}\|_{\mathbf{r},\rho}$$

Hence we can restrict our search for possible cycles of $\tau_{\mathbf{r}}$ to the finite set of witnesses

$$W_{\mathbf{r},\rho} := \left\{\mathbf{x} \in \mathbb{Z}^2 \mid \|\mathbf{x}\|_{\mathbf{r},\rho} \le \frac{\|(0,\ldots,0,1)\|_{\mathbf{r},\rho}}{\rho - 1}\right\}$$

which is the basis of the single parameter version of the algorithm.

To settle whole convex regions of $\mathbb{R}^2$ we observe that the norm $\|\cdot\|_{\mathbf{r},\rho}$ depends continuously on $\mathbf{r}$ and thus

$$\|\tau_{\mathbf{s}}(\mathbf{x})\|_{\mathbf{r},\rho} > \|\mathbf{x}\|_{\mathbf{r},\rho}$$

also holds for all $\mathbf{x} \in \mathbb{Z}^2$ and all $\mathbf{s} \in \mathbb{R}^2$ sufficiently close to $\mathbf{r}$. Thus there is a bounded set $K \subseteq \mathbb{R}^2$ of such $\mathbf{s}$ which contains $\mathbf{r}$ as an interior point and which has positive distance from the boundary of $\mathcal{D}_2$ (i.e. the SRS of all parameters in K are strictly expanding). We consider the following equivalence relation on K:

$$\mathbf{s} \sim \mathbf{t} \Leftrightarrow \forall\, \mathbf{x} \in W_{\mathbf{r},\rho} : \tau_{\mathbf{s}}(\mathbf{x}) = \tau_{\mathbf{t}}(\mathbf{x})$$

Since K is bounded and has positive distance form $\mathcal{D}_2$ it follows that $K/_\sim$ is a finite set and every element of $K/_\sim$ is either contained in $\mathcal{D}_2^{(*)}$ or has empty intersection with it by construction. Each of the finitely many parts can thus be settled by the single parameter version of the algorithm by taking an arbitrary parameter in the respective part. □

References

[1] S. Akiyama, T. Borbély, H. Brunotte, A. Pethő, and J. M. Thuswaldner, *Generalized radix representations and dynamical systems. I*, Acta Math. Hungar., 108 (2005), pp. 207–238.

[2] S. Akiyama, H. Brunotte, A. Pethő, and W. Steiner, *Remarks on a conjecture on certain integer sequences*, Period. Math. Hungar., 52 (2006), pp. 1–17.

[3] ———, *Periodicity of certain piecewise affine planar maps*, Tsukuba J. Math., 32 (2008), pp. 197–251.

[4] S. Akiyama, H. Brunotte, A. Pethő, and J. M. Thuswaldner, *Generalized radix representations and dynamical systems. II*, Acta Arith., 121 (2006), pp. 21–61.

[5] S. Akiyama, J.-H. Evertse, and A. Pethő, *On a binet-type formula for nearly linear recursive sequences and its applications*, in Number Theory - Diophantine problems, uniform distribution and applications, Festschrift in honour of Robert F. Tichy's 60th birthday, C. Elsholtz and P. Grabner, eds., Springer Verlag, 2017. To appear.

[6] S. Akiyama and A. Pethő, *Discretized rotation has infinitely many periodic orbits*, Nonlinearity, 26 (2013), pp. 871–880.

[7] P. Kirschenhofer, A. Pethő, P. Surer, and J. Thuswaldner, *Finite and periodic orbits of shift radix systems*, J. Théor. Nombres Bordeaux, 22 (2010), pp. 421–448.

[8] P. Kirschenhofer, A. Pethő, and J. M. Thuswaldner, *On a family of three term nonlinear integer recurrences*, Int. J. Number Theory, 4 (2008), pp. 135–146.

[9] P. Kirschenhofer and J. M. Thuswaldner, *Shift radix systems—a survey*, in Numeration and substitution 2012, RIMS Kôkyûroku Bessatsu, B46, Res. Inst. Math. Sci. (RIMS), Kyoto, 2014, pp. 1–59.

[10] J. C. Lagarias and Y. Wang, *Self-affine tiles in Rˆn*, Adv. Math., 121 (1996), pp. 21–49.

[11] I. Schur, *Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind. II*, J. reine und angew. Math, 148 (1918), pp. 122–145.

[12] M. Weitzer, *Characterization algorithms for shift radix systems with finiteness property*, Int. J. Number Theory, 11 (2015), pp. 211–232.

Attila Pethő, Faculty of Informatics, University of Debrecen, Kassai út 26, 4028 Debrecen, HUNGARY and, University of Ostrava, Faculty of Science, Dvořákova 7, 70103 Ostrava, Czech Republik

E-mail address: petho.attila@inf.unideb.hu

Jörg Thuswaldner, Chair of Mathematics and Statistics, University of Leoben, Franz Josef-Strasse 18, 8700 Leoben, AUSTRIA

E-mail address: joerg.thuswaldner@unileoben.ac.at

Mario Weitzer, Institute of Analysis and Number Theory, Graz University of Technology, Steyrergasse 30, 8010 Graz, AUSTRIA

E-mail address: weitzer@math.tugraz.at