An introduction to p-adic systems: A new kind of number system

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Advanced topics in discrete mathematics

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Motivation

Let $T_C : \mathbb{N} \to \mathbb{N}$

$$n \mapsto \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \mod 2 \\ \frac{3n+1}{2} & \text{if } n \equiv 1 \mod 2 \end{cases}$$

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Question for ultimate behaviour: Trivial!

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$(17, 8, 4, 2, 1, 0, \ldots) \,\%\, 2 = (1, 0, 0, 0, 1, 0, \ldots)$

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Notation: $S(T_C)[17] = (17, 26, 13, 20, \ldots)$: T_C-sequence of 17
$D(T_C)[17] = (1, 0, 1, 0, 0, 1, \ldots)$: T_C-(digit) expansion of 17

What do T_C and T_2 have in common?

Tables of sequences:

1	1	0	0	0	$\cdots$
2	2	1	0	0	$\cdots$
3	3	1	0	0	$\cdots$
4	4	2	1	0	$\cdots$
5	5	2	1	0	$\cdots$
6	6	3	1	0	$\cdots$
7	7	3	1	0	$\cdots$
8	8	4	2	1	$\cdots$
9	9	4	2	1	$\cdots$
10	10	5	2	1	$\cdots$
11	11	5	2	1	$\cdots$
12	12	6	3	1	$\cdots$
13	13	6	3	1	$\cdots$
14	14	7	3	1	$\cdots$
15	15	7	3	1	$\cdots$
16	16	8	4	2	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$S(T_2)$	0	1	2	3	$\cdots$

1	1	2	1	2	$\cdots$
2	2	1	2	1	$\cdots$
3	3	5	8	4	$\cdots$
4	4	2	1	2	$\cdots$
5	5	8	4	2	$\cdots$
6	6	3	5	8	$\cdots$
7	7	11	17	26	$\cdots$
8	8	4	2	1	$\cdots$
9	9	14	7	11	$\cdots$
10	10	5	8	4	$\cdots$
11	11	17	26	13	$\cdots$
12	12	6	3	5	$\cdots$
13	13	20	10	5	$\cdots$
14	14	7	11	17	$\cdots$
15	15	23	35	53	$\cdots$
16	16	8	4	2	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$S(T_C)$	0	1	2	3	$\cdots$

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11	1	1	0	1	$\cdots$
12	0	0	1	1	$\cdots$
13	1	0	1	1	$\cdots$
14	0	1	1	1	$\cdots$
15	1	1	1	1	$\cdots$
16	0	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$D(T_2)$	0	1	2	3	$\cdots$

1	1	0	1	0	$\cdots$
2	0	1	0	1	$\cdots$
3	1	1	0	0	$\cdots$
4	0	0	1	0	$\cdots$
5	1	0	0	0	$\cdots$
6	0	1	1	0	$\cdots$
7	1	1	1	0	$\cdots$
8	0	0	0	1	$\cdots$
9	1	0	1	1	$\cdots$
10	0	1	0	0	$\cdots$
11	1	1	0	1	$\cdots$
12	0	0	1	1	$\cdots$
13	1	0	0	1	$\cdots$
14	0	1	1	1	$\cdots$
15	1	1	1	1	$\cdots$
16	0	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$D(T_C)$	0	1	2	3	$\cdots$

What do T_C and T_2 have in common?

Tables of expansions:

1	1	0	0	0	$\cdots$
2	0	1	0	0	$\cdots$
3	1	1	0	0	$\cdots$
4	0	0	1	0	$\cdots$
5	1	0	1	0	$\cdots$
6	0	1	1	0	$\cdots$
7	1	1	1	0	$\cdots$
8	0	0	0	1	$\cdots$
9	1	0	0	1	$\cdots$
10	0	1	0	1	$\cdots$
11	1	1	0	1	$\cdots$
12	0	0	1	1	$\cdots$
13	1	0	1	1	$\cdots$
14	0	1	1	1	$\cdots$
15	1	1	1	1	$\cdots$
16	0	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
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5	1	0	0	0	$\cdots$
6	0	1	1	0	$\cdots$
7	1	1	1	0	$\cdots$
8	0	0	0	1	$\cdots$
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11	1	1	0	1	$\cdots$
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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$D(T_C)$	0	1	2	3	$\cdots$

First k digits of expansions of m and n coincide $\Leftrightarrow$ $m \equiv n \mod 2^k$

(Block property)

What do T_C and T_2 have in common?

What do T_C and T_2 have in common?

Both generalize to the 2-adic integers $\mathbb{Z}_2$:

$$T_C : \mathbb{Z}_2 \to \mathbb{Z}_2 \qquad n \mapsto \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \mod 2 \\ \frac{3n+1}{2} & \text{if } n \equiv 1 \mod 2 \end{cases}$$

$$T_2 : \mathbb{Z}_2 \to \mathbb{Z}_2 \qquad n \mapsto \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \mod 2 \\ \frac{n-1}{2} & \text{if } n \equiv 1 \mod 2 \end{cases}$$

p-adic numbers: the good/evil/equally likeable (?) twin of real numbers

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Real numbers $\mathbb{R}$	10-adic numbers $\mathbb{Q}_{10}$

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A crash course in p-adic numbers

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$\mathbb{Q} \subseteq \mathbb{R}$	

A crash course in p-adic numbers

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$\mathbb{Q} \subseteq \mathbb{R}$	$\mathbb{Q} \subseteq \mathbb{Q}_{10}$: $\overline{857142}9 \cdot 7 = 3 \Rightarrow 3/7 \in \mathbb{Q}_{10}$

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Completion of $\mathbb{Q}$ w.r.t. $\lvert\cdot\rvert$	Completion of $\mathbb{Q}$ w.r.t. $\lVert\cdot\rVert_{10}$

What do T_C and T_2 have in common?

Both generalize to the 2-adic integers $\mathbb{Z}_2$:

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Subtract LSD before division by p if necessary: $(x, x, x)(5) = \frac{5-2}{3} = 1$

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7 & 1 & 1 & 1 & 0 & \cdots \\
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\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\hline
D & 0 & 1 & 2 & 3 & \cdots
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\quad \longrightarrow \quad T(6) = 3 \quad \longrightarrow \quad T[0](6) \in \underbrace{\{6, 7\}}_{\text{remember: } (x, x-1) = (x, x)}
$$

Let $T = (T[0], \dots, T[p-1])$ be a p-adic system $(T[r] : \mathbb{Z}_p \to \mathbb{Z}_p)$

- $D(T)$ defines a bijection between $\mathbb{Z}_p$ and $\{0, \dots, p-1\}^{\mathbb{N}_0}$:
  the expansions of all p-adic integers are unique and every possible expansion occurs
- $D(T)$ is uniquely determined by the expansions of the natural numbers:
  $D(T)(n)[0, k-1] = D(T)(n \,\%\, p^k)[0, k-1]$ for all $n \in \mathbb{Z}_p$ $\quad$ ($\mathbb{N}$ dense in $\mathbb{Z}_p$)
- The sets of "p-digit tables" and p-adic systems are in one-to-one correspondence:
  If $D \in \{0, \dots, p-1\}^{\mathbb{N}_0 \times \mathbb{Z}_p}$ with
  - $D[n][0] = n \,\%\, p$
  - $D[m][0, k-1] = D[n][0, k-1] \quad \Leftrightarrow \quad m \equiv n \mod p^k$ (block property),

  then $D = D(T)$ for a "unique" p-adic system T

$$\begin{array}{c|ccccc}
1 & 1 & 0 & 1 & 0 & \cdots \\
2 & 0 & 1 & 0 & 1 & \cdots \\
3 & 1 & 1 & 0 & 0 & \cdots \\
4 & 0 & 0 & 1 & 0 & \cdots \\
5 & 1 & 0 & 0 & 0 & \cdots \\
6 & 0 & 1 & 1 & 0 & \cdots \\
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- One can define a group structure on the set of all p-adic systems

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Consequence: Yes, many exist!

More examples of p-adic system

- $T_n = (x, \dots, x)$, $T_C = (x, 3x+1)$

$$
\begin{array}{c|cccc}
\vdots & \vdots & \vdots & \vdots & \iddots \\
1 & 1 & 2 & 1 & \cdots \\
2 & 2 & 1 & 2 & \cdots \\
3 & 3 & 5 & 8 & \cdots \\
4 & 4 & 2 & 1 & \cdots \\
5 & 5 & 8 & 4 & \cdots \\
6 & 6 & 3 & 5 & \cdots \\
7 & 7 & 11 & 17 & \cdots \\
8 & 8 & 4 & 2 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\hline
S(T_C) & 0 & 1 & 2 & \cdots
\end{array}
\qquad
\begin{array}{c|cccc}
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1 & 1 & 0 & 1 & \cdots \\
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3 & 1 & 1 & 0 & \cdots \\
4 & 0 & 0 & 1 & \cdots \\
5 & 1 & 0 & 0 & \cdots \\
6 & 0 & 1 & 1 & \cdots \\
7 & 1 & 1 & 1 & \cdots \\
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$$

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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
1	1	34/11	64805/2541	$\cdots$
2	2	227/21	2053/231	$\cdots$
3	3	47/11	608/121	$\cdots$
4	4	880/21	12621386/3087	$\cdots$
5	5	60/11	64376/847	$\cdots$
6	6	639/7	4346/77	$\cdots$
7	7	73/11	777/121	$\cdots$
8	8	3338/21	59723107/1029	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$S(T)$	0	1	2	$\cdots$

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
1	1	0	1	$\cdots$
2	0	1	1	$\cdots$
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- $(\mathrm{i}x^2 - 6x + 5\mathrm{i}, x, x, x, x)$ where $\mathrm{i}^2 = -1$, i.e. $\mathrm{i} = \dots 31212$ or $\mathrm{i} = \dots 13233$

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  Example: $P(x) = 10x^2 - 3x + 4$ is a 2-permutation polynomial

$$
\begin{array}{c|r}
\vdots & \vdots \\
1 & 11 \\
2 & 38 \\
3 & 85 \\
4 & 152 \\
5 & 239 \\
6 & 346 \\
7 & 473 \\
8 & 620 \\
\vdots & \vdots \\
\hline
P &
\end{array}
\qquad
\begin{array}{c|cccc}
\vdots & \vdots & \vdots & \vdots & \iddots \\
1 & 1 & 1 & 0 & \cdots \\
2 & 0 & 1 & 1 & \cdots \\
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Digit period ($D(T_C)$)	Sequence period ($S(T_C)$)
0	0
$1, 0$	$1, 2$
1	-1
$1, 1, 0$	$-5, -7, -10$
$1, 1, 1, 1, 0, 1, 1, 1, 0, 0, 0$	$-17, -25, -37, -55, -82, -41, -61, -91, -136, -68, -34$

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  Note: $S(F)[n]$ ultimately periodic $\Leftrightarrow$ $D(F)[n]$ ultimately periodic

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For $p = 2$:

- $D(a_0 + xb_0, a_1 + xb_1)[n] = D(0 + xb_0, 1 + xb_1)[\frac{n(b_0-2)+a_0}{a_1(b_0-2)-a_0(b_1-2)}]$ for all $n \in \mathbb{Z}_2$

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  - All rational numbers have ultimately periodic expansions $\Leftrightarrow$ $b_0 \cdots b_{p-1} < p^p$
  - Expansions of integers admit only finitely many different periods

For $p = 2$:

- $D(a_0 + xb_0, a_1 + xb_1)[n] = D(0 + xb_0, 1 + xb_1)[\frac{n(b_0-2)+a_0}{a_1(b_0-2)-a_0(b_1-2)}]$ for all $n \in \mathbb{Z}_2$

  In particular: Whether or not all rational numbers have ultimately periodic expansions does not depend on constant coefficients a_i

- There is a p-adic system for which all rational numbers (in $\mathbb{Z}_p$) have finite expansions
- There is a p-adic system for which the natural numbers have ultimately periodic expansions with pairwise different periods

(Things get arbitrarily neat or nasty)

For p-adic systems defined by linear polynomials with integer coefficients:

$T = (a_0 + b_0x, \ldots, a_{p-1} + b_{p-1}x)$, $a_i, b_i \in \mathbb{Z}$

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Thank you!

Extra: a nice application

Lemma: If $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is (p, r)-suitable, then so is $g : \mathbb{Z}_p \to \mathbb{Z}_p$, $x \mapsto f(x) + px$

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