Triangulations of simplotopes and a general formula for an arithmetic constant due to G. R. Everest

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Let K : Number field

S : Finite set of places of K, containing Archimedean ones

ω_K : Number of roots of unity of K

$\mathrm{Reg}_{K,S}$: S-regulator of K

$q \in \mathbb{R}_{>0}$

Motivation: An arithmetic constant

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$q \in \mathbb{R}_{>0}$

$n \in \mathbb{N}$

$s := |S| - 1$

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$u_{K,S}(n;q)$: Number of representations of algebraic integers α with $\left|N_{K/\mathbb{Q}}(\alpha)\right| \leq q$ that can be written as sums of exactly n S-units

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Theorem (Fuchs, Tichy, Ziegler 2009)

$$u_{K,S}(n;q) = \frac{c_{n-1,s}}{n!}\left(\frac{\omega_K \log(q)^s}{\mathrm{Reg}_{K,S}}\right)^{n-1} + o(\log(q)^{(n-1)s-1+\varepsilon}) \quad (q \to \infty)$$

$c_{n,s}$ is the volume of

$$P_{n,s} := \{(x_{1,1},\ldots,x_{n,s}) \in \mathbb{R}^{ns} \mid g_{n,s}(x_{1,1},\ldots,x_{n,s}) \leq 1\}$$

where

$$g_{n,s}\begin{pmatrix} x_{1,1} & \cdots & x_{1,s} \\ \vdots & & \vdots \\ x_{n,1} & \cdots & x_{n,s} \end{pmatrix} := \max\left\{\begin{matrix} 0 \\ x_{1,1} \\ \vdots \\ x_{n,1} \end{matrix}\right\} + \cdots + \max\left\{\begin{matrix} 0 \\ x_{1,s} \\ \vdots \\ x_{n,s} \end{matrix}\right\} +$$

$$\max\left\{\begin{matrix} 0 \\ -x_{1,1} - \cdots - x_{1,s} \\ \vdots \\ -x_{n,1} - \cdots - x_{n,s} \end{matrix}\right\}$$

Note: Identify $\mathbb{R}^{ns}$ and $\mathbb{R}^{n \times s}$

A family of convex polytopes

$n\backslash s$	1	2	3	4	5
1	2	3	10/3	35/12	21/10
2	3	15/4	7/3	55/64	
3	4	7/2	55/54		
4	5	45/16			
5	6				

Table: Values of $c_{n,s} = \lambda_{ns}(P_{n,s})$

Previous results by Barroero, Frei, Fuchs, Tichy, and Ziegler:

Formulas for $c_{n,1}$, $c_{n,2}$, $c_{1,s}$

$$g_{n,s}\begin{pmatrix} x_{1,1} & \dots & x_{1,s} \\ \vdots & & \vdots \\ x_{n,1} & \dots & x_{n,s} \end{pmatrix} := \max\left\{\begin{matrix} 0 \\ x_{1,1} \\ \vdots \\ x_{n,1} \end{matrix}\right\} + \cdots + \max\left\{\begin{matrix} 0 \\ x_{1,s} \\ \vdots \\ x_{n,s} \end{matrix}\right\} +$$

$$\max\left\{\begin{matrix} 0 \\ -x_{1,1} - \cdots - x_{1,s} \\ \vdots \\ -x_{n,1} - \cdots - x_{n,s} \end{matrix}\right\}$$

$$P_{n,s} := \{(x_{1,1}, \dots, x_{n,s}) \in \mathbb{R}^{ns} \mid g_{n,s}(x_{1,1}, \dots, x_{n,s}) \leq 1\}$$

$$g_{n,s}\begin{pmatrix} x_{1,1} & \dots & x_{1,s} \\ \vdots & & \vdots \\ x_{n,1} & \dots & x_{n,s} \end{pmatrix} := \max\left\{\begin{matrix} 0 \\ x_{1,1} \\ \vdots \\ x_{n,1} \end{matrix}\right\} + \cdots + \max\left\{\begin{matrix} 0 \\ x_{1,s} \\ \vdots \\ x_{n,s} \end{matrix}\right\} + \max\left\{\begin{matrix} 0 \\ -x_{1,1} - \cdots - x_{1,s} \\ \vdots \\ -x_{n,1} - \cdots - x_{n,s} \end{matrix}\right\}$$

$$P_{n,s} := \{(x_{1,1}, \dots, x_{n,s}) \in \mathbb{R}^{ns} \mid g_{n,s}(x_{1,1}, \dots, x_{n,s}) \leq 1\}$$

$P_{n,s}$ is a

- closed non-degenerate convex polytope
- of dimension ns
- contained in $[-1, 1]^{ns}$
- with boundary $\partial(P_{n,s}) = \{\mathbf{x} \in \mathbb{R}^{ns} \mid g_{n,s}(\mathbf{x}) = 1\}$

A family of convex polytopes

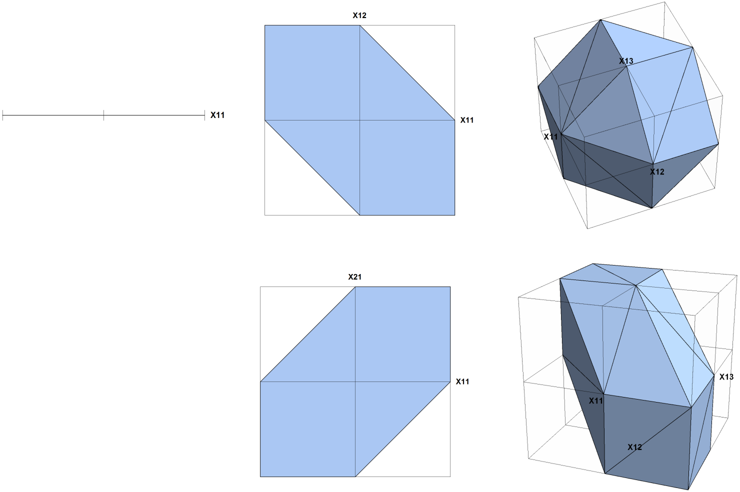

Figure: $P_{1,1}$, $P_{1,2}$, $P_{1,3}$, $P_{2,1}$, $P_{3,1}$

Main theorem (Kerber, Tichy, W.)

$$c_{n,s} = \frac{1}{(s!)^{n+1}} \frac{((n+1)s)!}{(ns)!}$$

for all $n, s \in \mathbb{N}$

Theorem

Let $n, s \in \mathbb{N}$ and $\mathcal{V}(P_{n,s})$ the set of vertices of $P_{n,s}$. Then

$$\mathcal{V}(P_{n,s}) = U_{n,s} \cup V_{n,s}$$

where

A family of convex polytopes: Vertices

Theorem

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where

$$U_{n,s} := \left\{ \mathbf{x} \in \{-1,0\}^{n \times s} \mid \begin{array}{l} \bullet \text{ at least one '}-1\text{'} \\ \bullet \text{ at most one '}-1\text{' per row} \end{array} \right\}$$

Theorem

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$$V_{n,s} := \Big\{ \mathbf{x} \in \{-1, 0, 1\}^{n \times s} \mid \begin{array}{l} \bullet \text{ at least one '1'} \\ \bullet \text{ all '1's in a single column (the "'1'-column")} \\ \bullet \text{ all entries of the '1'-column are '0' or '1'} \\ \bullet \text{ all rows with a '1' contain at most one '}-1\text{'} \\ \bullet \text{ all rows without a '1' contain only '0's} \end{array} \Big\}$$

Theorem

Let $n, s \in \mathbb{N}$ and $\mathcal{V}(P_{n,s})$ the set of vertices of $P_{n,s}$. Then

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In particular: $P_{n,s} = \operatorname{conv}(U_{n,s} \cup V_{n,s})$

$$U_{n,s} \;:=\; \left\{\mathbf{x} \in \{-1,0\}^{n\times s} \mid \begin{array}{l}\bullet \text{ at least one '}-1\text{'} \\ \bullet \text{ at most one '}-1\text{' per row}\end{array}\right\}$$

Example: $n = 2$ and $s = 3$

$$U_{2,3} = \Big\{$$

$$\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\Big\}$$

A family of convex polytopes: Vertices

$V_{n,s} \;:=\; \Big\{ \mathbf{x} \in \{-1,0,1\}^{n\times s} \;\Big|$
- at least one '1'
- all '1's in a single column (the "'1'-column")
- all entries of the '1'-column are '0' or '1'
- all rows with a '1' contain at most one '-1'
- all rows without a '1' contain only '0's$\Big\}$

Example: $n=2$ and $s=3$

$$
\begin{aligned}
V_{2,3} = \Big\{ & \\
& \begin{pmatrix} 1&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 1&-1&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&-1\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\1&-1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\1&0&-1 \end{pmatrix}, \\
& \begin{pmatrix} 1&0&0\\1&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0\\1&-1&0 \end{pmatrix}, \begin{pmatrix} 1&0&0\\1&0&-1 \end{pmatrix}, \begin{pmatrix} 1&-1&0\\1&0&0 \end{pmatrix}, \begin{pmatrix} 1&-1&0\\1&-1&0 \end{pmatrix}, \begin{pmatrix} 1&-1&0\\1&0&-1 \end{pmatrix}, \begin{pmatrix} 1&0&-1\\1&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&-1\\1&-1&0 \end{pmatrix}, \begin{pmatrix} 1&0&-1\\1&0&-1 \end{pmatrix} \\
& \begin{pmatrix} 0&1&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} -1&1&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&-1\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\-1&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&1&-1 \end{pmatrix}, \\
& \begin{pmatrix} 0&1&0\\0&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&0\\-1&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&0\\0&1&-1 \end{pmatrix}, \begin{pmatrix} -1&1&0\\0&1&0 \end{pmatrix}, \begin{pmatrix} -1&1&0\\-1&1&0 \end{pmatrix}, \begin{pmatrix} -1&1&0\\0&1&-1 \end{pmatrix}, \begin{pmatrix} 0&1&-1\\0&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&-1\\-1&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&-1\\0&1&-1 \end{pmatrix} \\
& \begin{pmatrix} 0&0&1\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&-1&1\\0&0&0 \end{pmatrix}, \begin{pmatrix} -1&0&1\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&-1&1 \end{pmatrix}, \begin{pmatrix} 0&0&0\\-1&0&1 \end{pmatrix}, \\
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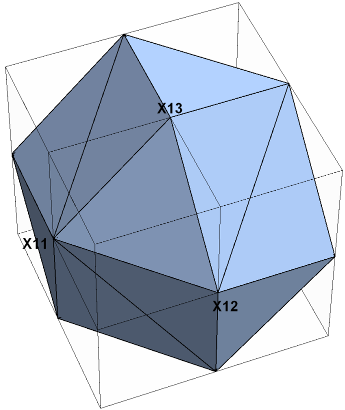
\Big\} &
\end{aligned}
$$



Corollary

$c_{n,s}$ is integer multiple of $1/(ns)!$

$1/(ns)!$: Volume of standard simplex $\mathrm{conv}(\{\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_{ns}\})$

$n \backslash s$	1	2	3	4
1	2	3	$10/3$	$35/12$
2	3	$15/4$	$7/3$	$55/64$
3	4	$7/2$	$55/54$	
4	5	$45/16$		

Table: Values of $c_{n,s} = \lambda_{ns}(P_{n,s})$

$n \backslash s$	1	2	3	4
1	2	6	20	70
2	6	90	1680	34650
3	24	2520	369600	
4	120	113400		

Table: Values of $c_{n,s}(ns)!$

$n\backslash s$	1	2	3	4
1	2	3	$10/3$	$35/12$
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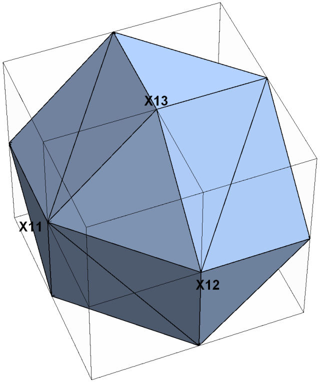
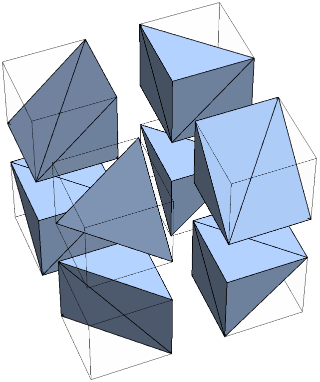
$n\backslash s$	1	2	3	4
1	2	6	20	70
2	6	90	1680	34650
3	24	2520	369600	
4	120	113400		

Table: Values of $c_{n,s}(ns)!$

Reminder: $c_{n,s} = \frac{1}{(s!)^{n+1}} \frac{((n+1)s)!}{(ns)!} = \binom{(n+1)s}{s,\ldots,s} \frac{1}{(ns)!}$

A family of convex polytopes: A closer look





Corollary

$P_{n,s}$ is the disjoint union of 2^{ns} smaller convex polytopes

Introduce additional vertex $\mathbf{0} \in \mathrm{int}(P_{n,s})$

A family of convex polytopes: A closer look

“Normed volumes” of the 2^{ns} parts?

$n = 1$, $s = 1$:
$\leq$: 1
$\geq$: 1

Figure: $P_{1,1}$

A family of convex polytopes: A closer look

"Normed volumes" of the 2^{ns} parts?

$n = 1$, $s = 1$:
$\leq$: 1
$\geq$: 1
$n = 1$, $s = 2$:
$\leq\leq$: 2
$\leq\geq$: 2
$\geq\leq$: 2
$\geq\geq$: 1

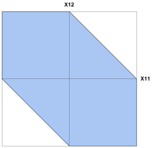


Figure: $P_{1,2}$

"Normed volumes" of the 2^{ns} parts?

$n = 1$, $s = 1$:
$\leq$: 1
$\geq$: 1
$n = 1$, $s = 2$:
$\leq\leq$: 1
$\leq\geq$: 2
$\geq\leq$: 2
$\geq\geq$: 1
$n = 1$, $s = 3$:
$\leq\leq\leq$: 1
$\leq\leq\geq$: 3
$\leq\geq\leq$: 3
$\leq\geq\geq$: 3
$\geq\leq\leq$: 3
$\geq\leq\geq$: 3
$\geq\geq\leq$: 3
$\geq\geq\geq$: 1

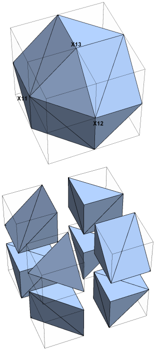


Figure: $P_{1,3}$

"Normed volumes" of the 2^{ns} parts?

$n = 1,\ s = 1$:
$\leq$: 1
$\geq$: 1

$n = 1,\ s = 2$:
$\leq\leq$: 1
$\leq\geq$: 2
$\geq\leq$: 2
$\geq\geq$: 1

$n = 1,\ s = 3$:
$\leq\leq\leq$: 1
$\leq\leq\geq$: 3
$\leq\geq\leq$: 3
$\leq\geq\geq$: 3
$\geq\leq\leq$: 3
$\geq\leq\geq$: 3
$\geq\geq\leq$: 3
$\geq\geq\geq$: 1

$n = 2,\ s = 1$:
$\leq\leq$: 2
$\leq\geq$: 1
$\geq\leq$: 1
$\geq\geq$: 2

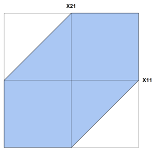


Figure: $P_{2,1}$

A family of convex polytopes: A closer look

"Normed volumes" of the 2^{ns} parts?

$n = 1$, $s = 1$:
$\leq$: 1
$\geq$: 1
$n = 1$, $s = 2$:
$\leq\leq$: 1
$\leq\geq$: 2
$\geq\leq$: 2
$\geq\geq$: 1
$n = 1$, $s = 3$:
$\leq\leq\leq$: 1
$\leq\leq\geq$: 3
$\leq\geq\leq$: 3
$\leq\geq\geq$: 3
$\geq\leq\leq$: 3
$\geq\leq\geq$: 3
$\geq\geq\leq$: 3
$\geq\geq\geq$: 1

$n = 2$, $s = 1$:
$\leq\leq$: 2
$\leq\geq$: 1
$\geq\leq$: 1
$\geq\geq$: 2
$n = 2$, $s = 2$:
$\leq\leq\leq\leq$: 6
$\leq\leq\leq\geq$: 4
$\leq\leq\geq\leq$: 4
$\leq\leq\geq\geq$: 1
$\vdots$

"Normed volumes" of the 2^{ns} parts?

$n = 1,\ s = 1$:
$\leq$: 1
$\geq$: 1

$n = 1,\ s = 2$:
$\leq\leq$: 1
$\leq\geq$: 2
$\geq\leq$: 2
$\geq\geq$: 1

$n = 1,\ s = 3$:
$\leq\leq\leq$: 1
$\leq\leq\geq$: 3
$\leq\geq\leq$: 3
$\leq\geq\geq$: 3
$\geq\leq\leq$: 3
$\geq\leq\geq$: 3
$\geq\geq\leq$: 3
$\geq\geq\geq$: 1

$n = 2,\ s = 1$:
$\leq\leq$: 2
$\leq\geq$: 1
$\geq\leq$: 1
$\geq\geq$: 2

$n = 2,\ s = 2$:
$\leq\leq\leq\leq$: 6
$\leq\leq\leq\geq$: 4
$\leq\leq\geq\leq$: 4
$\leq\leq\geq\geq$: 1
$\vdots$

$n = 2,\ s = 3$:
$\leq\leq\leq\leq\leq\leq$: 20
$\leq\leq\leq\leq\leq\geq$: 15
$\leq\leq\leq\leq\geq\leq$: 15
$\leq\leq\leq\leq\geq\geq$: 6
$\vdots$

A family of convex polytopes: A closer look

"Normed volumes" of the 2^{ns} parts?

$n = 1,\ s = 1$:
$\leq$: 1
$\geq$: 1
$n = 1,\ s = 2$:
$\leq\leq$: 1
$\leq\geq$: 2
$\geq\leq$: 2
$\geq\geq$: 1
$n = 1,\ s = 3$:
$\leq\leq\leq$: 1
$\leq\leq\geq$: 3
$\leq\geq\leq$: 3
$\leq\geq\geq$: 3
$\geq\leq\leq$: 3
$\geq\leq\geq$: 3
$\geq\geq\leq$: 3
$\geq\geq\geq$: 1

$n = 2,\ s = 1$:
$\leq\leq$: 2
$\leq\geq$: 1
$\geq\leq$: 1
$\geq\geq$: 2
$n = 2,\ s = 2$:
$\leq\leq\leq\leq$: 6
$\leq\leq\leq\geq$: 4
$\leq\leq\geq\leq$: 4
$\leq\leq\geq\geq$: 1
$\vdots$
$n = 2,\ s = 3$:
$\leq\leq\leq\leq\leq\leq$: 20
$\leq\leq\leq\leq\leq\geq$: 15
$\leq\leq\leq\leq\geq\leq$: 15
$\leq\leq\leq\leq\geq\geq$: 6
$\vdots$

$n = 3,\ s = 1$:
$\leq\leq\leq$: 6
$\leq\leq\geq$: 2
$\leq\geq\leq$: 2
$\leq\geq\geq$: 2
$\geq\leq\leq$: 2
$\geq\leq\geq$: 2
$\geq\geq\leq$: 2
$\geq\geq\geq$: 6
$\vdots$

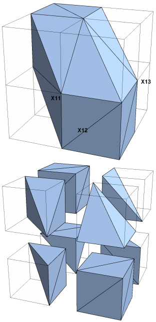


Figure: $P_{3,1}$

"Normed volumes" of the 2^{ns} parts?

$n = 1,\ s = 1$:
$\leq$: 1
$\geq$: 1
$n = 1,\ s = 2$:
$\leq\leq$: 1
$\leq\geq$: 2
$\geq\leq$: 2
$\geq\geq$: 1
$n = 1,\ s = 3$:
$\leq\leq\leq$: 1
$\leq\leq\geq$: 3
$\leq\geq\leq$: 3
$\leq\geq\geq$: 3
$\geq\leq\leq$: 3
$\geq\leq\geq$: 3
$\geq\geq\leq$: 3
$\geq\geq\geq$: 1

$n = 2,\ s = 1$:
$\leq\leq$: 2
$\leq\geq$: 1
$\geq\leq$: 1
$\geq\geq$: 2
$n = 2,\ s = 2$:
$\leq\leq\leq\leq$: 6
$\leq\leq\leq\geq$: 4
$\leq\leq\geq\leq$: 4
$\leq\leq\geq\geq$: 1
$\vdots$
$n = 2,\ s = 3$:
$\leq\leq\leq\leq\leq\leq$: 20
$\leq\leq\leq\leq\leq\geq$: 15
$\leq\leq\leq\leq\geq\leq$: 15
$\leq\leq\leq\leq\geq\geq$: 6
$\vdots$

$n = 3,\ s = 1$:
$\leq\leq\leq$: 6
$\leq\leq\geq$: 2
$\leq\geq\leq$: 2
$\leq\geq\geq$: 2
$\vdots$
$n = 3,\ s = 2$:
$\leq\leq\leq\leq\leq\leq$: 90
$\leq\leq\leq\leq\leq\geq$: 36
$\leq\leq\leq\leq\geq\leq$: 36
$\leq\leq\leq\leq\geq\geq$: 6
$\vdots$

A family of convex polytopes: A closer look

"Normed volumes" of the 2^{ns} parts?

$n = 1,\ s = 1$:
$\leq$: 1
$\geq$: 1

$n = 1,\ s = 2$:
$\leq\leq$: 1
$\leq\geq$: 2
$\geq\leq$: 2
$\geq\geq$: 1

$n = 1,\ s = 3$:
$\leq\leq\leq$: 1
$\leq\leq\geq$: 3
$\leq\geq\leq$: 3
$\leq\geq\geq$: 3
$\geq\leq\leq$: 3
$\geq\leq\geq$: 3
$\geq\geq\leq$: 3
$\geq\geq\geq$: 1

$n = 2,\ s = 1$:
$\leq\leq$: 2
$\leq\geq$: 1
$\geq\leq$: 1
$\geq\geq$: 2

$n = 2,\ s = 2$:
$\leq\leq\leq\leq$: 6
$\leq\leq\leq\geq$: 4
$\leq\leq\geq\leq$: 4
$\leq\leq\geq\geq$: 1
$\vdots$

$n = 2,\ s = 3$:
$\leq\leq\leq\leq\leq\leq$: 20
$\leq\leq\leq\leq\leq\geq$: 15
$\leq\leq\leq\leq\geq\leq$: 15
$\leq\leq\leq\leq\geq\geq$: 6
$\vdots$

$n = 3,\ s = 1$:
$\leq\leq\leq$: 6
$\leq\leq\geq$: 2
$\leq\geq\leq$: 2
$\leq\geq\geq$: 2
$\vdots$

$n = 3,\ s = 2$:
$\leq\leq\leq\leq\leq\leq$: 90
$\leq\leq\leq\leq\leq\geq$: 36
$\leq\leq\leq\leq\geq\leq$: 36
$\leq\leq\leq\leq\geq\geq$: 6
$\vdots$

$n = 3,\ s = 3$:
$\leq\leq\leq\leq\leq\leq\leq\leq\leq$: 1680
$\leq\leq\leq\leq\leq\leq\leq\leq\geq$: 720
$\leq\leq\leq\leq\leq\leq\leq\geq\leq$: 720
$\leq\leq\leq\leq\leq\leq\leq\geq\geq$: 180
$\vdots$

A family of convex polytopes: A closer look

"Normed volumes" of the 2^{ns} parts?

$n=1,\ s=1$:
$\leq$: 1
$\geq$: 1
$n=1,\ s=2$:
$\leq\leq$: 1
$\leq\geq$: 2
$\geq\leq$: 2
$\geq\geq$: 1
$n=1,\ s=3$:
$\leq\leq\leq$: 1
$\leq\leq\geq$: 3
$\leq\geq\leq$: 3
$\leq\geq\geq$: 3
$\geq\leq\leq$: 3
$\geq\leq\geq$: 3
$\geq\geq\leq$: 3
$\geq\geq\geq$: 1

$n=2,\ s=1$:
$\leq\leq$: 2
$\leq\geq$: 1
$\geq\leq$: 1
$\geq\geq$: 2
$n=2,\ s=2$:
$\leq\leq\leq\leq$: 6
$\leq\leq\leq\geq$: 4
$\leq\leq\geq\leq$: 4
$\leq\leq\geq\geq$: 1
$\vdots$
$n=2,\ s=3$:
$\leq\leq\leq\leq\leq\leq$: 20
$\leq\leq\leq\leq\leq\geq$: 15
$\leq\leq\leq\leq\geq\leq$: 15
$\leq\leq\leq\leq\geq\geq$: 6
$\vdots$

$n=3,\ s=1$:
$\leq\leq\leq$: 6
$\leq\leq\geq$: 2
$\leq\geq\leq$: 2
$\leq\geq\geq$: 2
$\vdots$
$n=3,\ s=2$:
$\leq\leq\leq\leq\leq\leq$: 90
$\leq\leq\leq\leq\leq\geq$: 36
$\leq\leq\leq\leq\geq\leq$: 36
$\leq\leq\leq\leq\geq\geq$: 6
$\vdots$
$n=3,\ s=3$:
$\leq\leq\leq\leq\leq\leq\leq\leq\leq$: 1680
$\leq\leq\leq\leq\leq\leq\leq\leq\geq$: 720
$\leq\leq\leq\leq\leq\leq\leq\geq\leq$: 720
$\leq\leq\leq\leq\leq\leq\leq\geq\geq$: 180
$\vdots$

$n=1,\ s=1$:
2

$n=1,\ s=2$:
6

$n=1,\ s=3$:
20

$n=2,\ s=1$:
6

$n=2,\ s=2$:
90

$n=2,\ s=3$:
1680

"Normed volumes" of the 2^{ns} parts?

$n = 1,\ s = 1$:
$\leq$: 1
$\geq$: 1
$n = 1,\ s = 2$:
$\leq\leq$: 1
$\leq\geq$: 2
$\geq\leq$: 2
$\geq\geq$: 1
$n = 1,\ s = 3$:
$\leq\leq\leq$: 1
$\leq\leq\geq$: 3
$\leq\geq\leq$: 3
$\leq\geq\geq$: 3
$\geq\leq\leq$: 3
$\geq\leq\geq$: 3
$\geq\geq\leq$: 3
$\geq\geq\geq$: 1

$n = 2,\ s = 1$:
$\leq\leq$: 2
$\leq\geq$: 1
$\geq\leq$: 1
$\geq\geq$: 2
$n = 2,\ s = 2$:
$\leq\leq\leq\leq$: 6
$\leq\leq\leq\geq$: 4
$\leq\leq\geq\leq$: 4
$\leq\leq\geq\geq$: 1
$\vdots$
$n = 2,\ s = 3$:
$\leq\leq\leq\leq\leq\leq$: 20
$\leq\leq\leq\leq\leq\geq$: 15
$\leq\leq\leq\leq\geq\leq$: 15
$\leq\leq\leq\leq\geq\geq$: 6
$\vdots$

$n = 3,\ s = 1$:
$\leq\leq\leq$: 6
$\leq\leq\geq$: 2
$\leq\geq\leq$: 2
$\leq\geq\geq$: 2
$\vdots$
$n = 3,\ s = 2$:
$\leq\leq\leq\leq\leq\leq$: 90
$\leq\leq\leq\leq\leq\geq$: 36
$\leq\leq\leq\leq\geq\leq$: 36
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$\vdots$
$n = 3,\ s = 3$:
$\leq\leq\leq\leq\leq\leq\leq\leq\leq$: 1680
$\leq\leq\leq\leq\leq\leq\leq\leq\geq$: 720
$\leq\leq\leq\leq\leq\leq\leq\geq\leq$: 720
$\leq\leq\leq\leq\leq\leq\leq\geq\geq$: 180
$\vdots$

$n = 1,\ s = 1$:
2

$n = 1,\ s = 2$:
6

$n = 1,\ s = 3$:
20

$n = 2,\ s = 1$:
6

$n = 2,\ s = 2$:
90

$n = 2,\ s = 3$:
1680

A family of convex polytopes: A closer look

$n \backslash s$	1	2	3	4
1	2	6	20	70
2	6	90	1680	34650
3	24	2520	369600	
4	120	113400		

Table: Normed volumes of $P_{n,s}$

$n \backslash s$	1	2	3	4
1	1	1	1	1
2	2	6	20	70
3	6	90	1680	34650
4	24	2520	369600	

Table: Normed volumes of “bottom-left quadrant” of $P_{n,s}$

A family of convex polytopes: The "bottom-left quadrant"

Reminder: $\mathcal{V}(P_{n,s}) = U_{n,s} \cup V_{n,s}$ where

$$U_{n,s} := \Big\{ \mathbf{x} \in \{-1,0\}^{n\times s} \mid \begin{array}{l} \bullet \text{ at least one '}-1\text{'} \\ \bullet \text{ at most one '}-1\text{' per row} \end{array} \Big\}$$

$$V_{n,s} := \Big\{ \mathbf{x} \in \{-1,0,1\}^{n\times s} \mid \begin{array}{l} \bullet \text{ at least one '1'} \\ \bullet \text{ all '1's in a single column (the "'1'-column")} \\ \bullet \text{ all entries of the '1'-column are '0' or '1'} \\ \bullet \text{ all rows with a '1' contain at most one '}-1\text{'} \\ \bullet \text{ all rows without a '1' contain only '0's} \end{array} \Big\}$$

A family of convex polytopes: The "bottom-left quadrant"

Reminder: $\mathcal{V}(P_{n,s}) = U_{n,s} \cup V_{n,s}$ where

$$U_{n,s} := \Big\{ \mathbf{x} \in \{-1,0\}^{n\times s} \mid \begin{array}{l} \bullet \text{ at least one '}{-1}\text{'} \\ \bullet \text{ at most one '}{-1}\text{' per row} \end{array} \Big\}$$

$$V_{n,s} := \Big\{ \mathbf{x} \in \{-1,0,1\}^{n\times s} \mid \begin{array}{l} \bullet \text{ at least one '1'} \\ \bullet \text{ all '1's in a single column (the "'1'-column")} \\ \bullet \text{ all entries of the '1'-column are '0' or '1'} \\ \bullet \text{ all rows with a '1' contain at most one '}{-1}\text{'} \\ \bullet \text{ all rows without a '1' contain only '0's} \end{array} \Big\}$$

Bottom-left quadrant:

$$S_{n,s} := \operatorname{conv}(U_{n,s} \cup \{\mathbf{0}\})$$

A family of convex polytopes: The "bottom-left quadrant"

Reminder: $\mathcal{V}(P_{n,s}) = U_{n,s} \cup V_{n,s}$ where

$$U_{n,s} := \Big\{\mathbf{x} \in \{-1,0\}^{n\times s} \mid \begin{array}{l} \bullet \text{ at least one '}-1\text{'} \\ \bullet \text{ at most one '}-1\text{' per row} \end{array}\Big\}$$

$$V_{n,s} := \Big\{\mathbf{x} \in \{-1,0,1\}^{n\times s} \mid \begin{array}{l} \bullet \text{ at least one '1'} \\ \bullet \text{ all '1's in a single column (the ``'1'-column'')} \\ \bullet \text{ all entries of the '1'-column are '0' or '1'} \\ \bullet \text{ all rows with a '1' contain at most one '}-1\text{'} \\ \bullet \text{ all rows without a '1' contain only '0's} \end{array}\Big\}$$

Bottom-left quadrant:

$$\begin{aligned} S_{n,s} &:= \operatorname{conv}(U_{n,s} \cup \{\mathbf{0}\}) \\ &= \operatorname{conv}\left(\{\mathbf{x} \in \{-1,0\}^{n\times s} \mid \text{at most one '}-1\text{' per row}\}\right) \end{aligned}$$

A family of convex polytopes: The "bottom-left quadrant"

Reminder: $\mathcal{V}(P_{n,s}) = U_{n,s} \cup V_{n,s}$ where

$$U_{n,s} := \Big\{ \mathbf{x} \in \{-1,0\}^{n\times s} \mid \begin{array}{l} \bullet \text{ at least one '}-1\text{'} \\ \bullet \text{ at most one '}-1\text{' per row} \end{array} \Big\}$$

$$V_{n,s} := \Big\{ \mathbf{x} \in \{-1,0,1\}^{n\times s} \mid \begin{array}{l} \bullet \text{ at least one '1'} \\ \bullet \text{ all '1's in a single column (the "'1'-column")} \\ \bullet \text{ all entries of the '1'-column are '0' or '1'} \\ \bullet \text{ all rows with a '1' contain at most one '}-1\text{'} \\ \bullet \text{ all rows without a '1' contain only '0's} \end{array} \Big\}$$

Bottom-left quadrant:

$$\begin{aligned} S_{n,s} &:= \operatorname{conv}(U_{n,s} \cup \{\mathbf{0}\}) \\ &= \operatorname{conv}\left(\left\{ \mathbf{x} \in \{-1,0\}^{n\times s} \mid \text{at most one '}-1\text{' per row} \right\} \right) \\ &= \operatorname{conv}\left(\left\{ \mathbf{x} \in \{-1,0\}^{s} \mid \text{at most one '}-1\text{'} \right\} \right)^{n} \text{ (Cartesian product)} \end{aligned}$$

A family of convex polytopes: The "bottom-left quadrant"

Reminder: $\mathcal{V}(P_{n,s}) = U_{n,s} \cup V_{n,s}$ where

$$U_{n,s} := \Big\{ \mathbf{x} \in \{-1,0\}^{n\times s} \mid \bullet \text{ at least one '}-1\text{'} \quad \bullet \text{ at most one '}-1\text{' per row} \Big\}$$

$$V_{n,s} := \Big\{ \mathbf{x} \in \{-1,0,1\}^{n\times s} \mid \bullet \text{ at least one '1'} \quad \bullet \text{ all '1's in a single column (the "'1'-column")} \quad \bullet \text{ all entries of the '1'-column are '0' or '1'} \quad \bullet \text{ all rows with a '1' contain at most one '}-1\text{'} \quad \bullet \text{ all rows without a '1' contain only '0's} \Big\}$$

Bottom-left quadrant:

$$\begin{aligned} S_{n,s} &:= \operatorname{conv}(U_{n,s} \cup \{\mathbf{0}\}) \\ &= \operatorname{conv}\left(\left\{\mathbf{x} \in \{-1,0\}^{n\times s} \mid \text{at most one '}-1\text{' per row}\right\}\right) \\ &= \operatorname{conv}\left(\left\{\mathbf{x} \in \{-1,0\}^{s} \mid \text{at most one '}-1\text{'}\right\}\right)^n \text{ (Cartesian product)} \\ &= \operatorname{conv}\left(\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\}\right)^n \end{aligned}$$

A family of convex polytopes: The "bottom-left quadrant"

$S_{n,s} = \operatorname{conv}\left(\left\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\right\}\right)^n$

A family of convex polytopes: The "bottom-left quadrant"

$S_{n,s} = \operatorname{conv}\left(\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\}\right)^n$

$\operatorname{conv}\left(\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\}\right)$: s-dimensional simplex

A family of convex polytopes: The "bottom-left quadrant"

$S_{n,s} = \mathrm{conv}\left(\left\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\right\}\right)^n$

$\mathrm{conv}\left(\left\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\right\}\right)$: s-dimensional simplex

Cartesian product of simplices: Simplotope

A family of convex polytopes: The "bottom-left quadrant"

$S_{n,s} = \text{conv}\left(\left\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\right\}\right)^n$

$\text{conv}\left(\left\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\right\}\right)$: s-dimensional simplex

Cartesian product of simplices: Simplotope

Special case $n = 1$: 1-fold product of simplex: s-dimensional simplex

A family of convex polytopes: The “bottom-left quadrant”

$S_{n,s} = \text{conv}\left(\left\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\right\}\right)^n$

$\text{conv}\left(\left\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\right\}\right)$: s-dimensional simplex

Cartesian product of simplices: Simplotope

Special case $n = 1$: 1-fold product of simplex: s-dimensional simplex
Special case $s = 1$: n-fold product of $[-1, 0]$: n-dimensional hypercube

A family of convex polytopes: The "bottom-left quadrant"

$S_{n,s} = \text{conv}\left(\left\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\right\}\right)^n$

$\text{conv}\left(\left\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\right\}\right)$: s-dimensional simplex

Cartesian product of simplices: Simplotope

Special case $n = 1$: 1-fold product of simplex: s-dimensional simplex
Special case $s = 1$: n-fold product of $[-1, 0]$: n-dimensional hypercube
Special case $s = 2$: n-fold product of $\text{conv}(\{(0,0),(-1,0),(0,-1)\})$

A family of convex polytopes: The "bottom-left quadrant"

$S_{n,s} = \text{conv}\left(\left\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\right\}\right)^n$

$\text{conv}\left(\left\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\right\}\right)$: s-dimensional simplex

Cartesian product of simplices: Simplotope

Special case $n = 1$: 1-fold product of simplex: s-dimensional simplex
Special case $s = 1$: n-fold product of $[-1, 0]$: n-dimensional hypercube
Special case $s = 2$: n-fold product of $\text{conv}(\{(0,0), (-1,0), (0,-1)\})$

Volume: $\lambda_{ns}(S_{n,s}) = \lambda_{ns}\left(\text{conv}\left(\left\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\right\}\right)^n\right)$

A family of convex polytopes: The "bottom-left quadrant"

$S_{n,s} = \text{conv}\left(\left\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\right\}\right)^n$

$\text{conv}\left(\left\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\right\}\right)$: s-dimensional simplex

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Special case $n = 1$: 1-fold product of simplex: s-dimensional simplex
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Volume: $\lambda_{ns}(S_{n,s}) = \lambda_{ns}\left(\text{conv}\left(\left\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\right\}\right)^n\right)$
$= \lambda_s\left(\text{conv}\left(\left\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\right\}\right)\right)^n$ (Fubini)

A family of convex polytopes: The "bottom-left quadrant"

$S_{n,s} = \mathrm{conv}\left(\left\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\right\}\right)^n$

$\mathrm{conv}\left(\left\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\right\}\right)$: s-dimensional simplex

Cartesian product of simplices: Simplotope

Special case $n = 1$: 1-fold product of simplex: s-dimensional simplex
Special case $s = 1$: n-fold product of $[-1, 0]$: n-dimensional hypercube
Special case $s = 2$: n-fold product of $\mathrm{conv}(\{(0,0), (-1,0), (0,-1)\})$

Volume: $$\begin{aligned}\lambda_{ns}(S_{n,s}) &= \lambda_{ns}\left(\mathrm{conv}\left(\left\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\right\}\right)^n\right)\\ &= \lambda_s\left(\mathrm{conv}\left(\left\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\right\}\right)\right)^n \text{ (Fubini)}\\ &= 1/(s!)^n\end{aligned}$$

A family of convex polytopes: The “bottom-left quadrant”

$S_{n,s} = \mathrm{conv}\left(\left\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\right\}\right)^n$

$\mathrm{conv}\left(\left\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\right\}\right)$: s-dimensional simplex

Cartesian product of simplices: Simplotope

Special case $n = 1$: 1-fold product of simplex: s-dimensional simplex
Special case $s = 1$: n-fold product of $[-1, 0]$: n-dimensional hypercube
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Volume:
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$\lambda_{ns}(P_{n,s})(ns)! = \lambda_{(n+1)s}(S_{n+1,s})((n+1)s)!$

A family of convex polytopes: The "bottom-left quadrant"

$S_{n,s} = \operatorname{conv}\left(\left\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\right\}\right)^n$

$\operatorname{conv}\left(\left\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\right\}\right)$: *s*-dimensional simplex

Cartesian product of simplices: Simplotope

Special case $n = 1$: 1-fold product of simplex: s-dimensional simplex
Special case $s = 1$: n-fold product of $[-1, 0]$: n-dimensional hypercube
Special case $s = 2$: n-fold product of $\operatorname{conv}(\{(0,0), (-1,0), (0,-1)\})$

Volume:
$$\begin{aligned}\lambda_{ns}(S_{n,s}) &= \lambda_{ns}\left(\operatorname{conv}\left(\left\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\right\}\right)^n\right)\\ &= \lambda_s\left(\operatorname{conv}\left(\left\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\right\}\right)\right)^n \text{ (Fubini)}\\ &= 1/(s!)^n\end{aligned}$$

$$\lambda_{ns}(P_{n,s})(ns)! = \lambda_{(n+1)s}(S_{n+1,s})((n+1)s)! = 1/(s!)^{n+1}((n+1)s)!$$

A family of convex polytopes: The "bottom-left quadrant"

$S_{n,s} = \mathrm{conv}\left(\left\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\right\}\right)^n$

$\mathrm{conv}\left(\left\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\right\}\right)$: s-dimensional simplex

Cartesian product of simplices: Simplotope

Special case $n = 1$: 1-fold product of simplex: s-dimensional simplex
Special case $s = 1$: n-fold product of $[-1, 0]$: n-dimensional hypercube
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Volume: $\lambda_{ns}(S_{n,s}) = \lambda_{ns}\left(\mathrm{conv}\left(\left\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\right\}\right)^n\right)$
$= \lambda_s\left(\mathrm{conv}\left(\left\{\mathbf{0}, -\mathbf{e}_1, \ldots, -\mathbf{e}_s\right\}\right)\right)^n$ (Fubini)
$= 1/(s!)^n$

$\lambda_{ns}(P_{n,s})(ns)! = \lambda_{(n+1)s}(S_{n+1,s})((n+1)s)! = 1/(s!)^{n+1}((n+1)s)!$

Thus $c_{n,s} = \lambda_{ns}(P_{n,s}) = \frac{1}{(s!)^{n+1}} \frac{((n+1)s)!}{(ns)!}$

A family of convex polytopes: The "bottom-left quadrant"

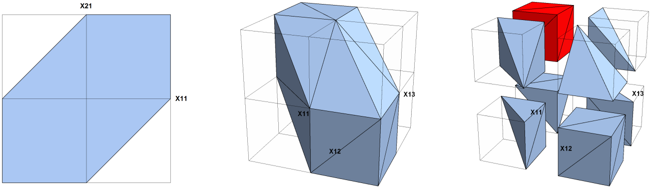

Figure: $P_{2,1}$ and $P_{3,1}$

Left to show: Normed volume of $P_{n,s}$ = Normed volume of $S_{n+1,s}$

A family of convex polytopes: The “bottom-left quadrant”

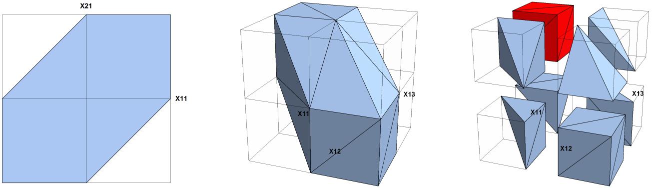

Figure: $P_{2,1}$ and $P_{3,1}$

Left to show: Normed volume of $P_{n,s}$ = Normed volume of $S_{n+1,s}$

Counting vertices: $|U_{n,s}| = (s+1)^n - 1$
$|V_{n,s}| = s((s+1)^n - 1)$

A family of convex polytopes: The "bottom-left quadrant"

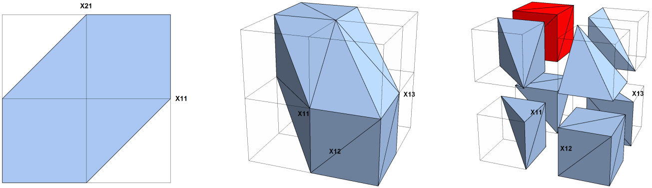

Figure: $P_{2,1}$ and $P_{3,1}$

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Counting vertices: $|U_{n,s}| = (s+1)^n - 1$
$|V_{n,s}| = s((s+1)^n - 1)$

$$|\mathcal{V}(P_{n,s})| = |U_{n,s}| + |V_{n,s}| = (s+1)^{n+1} - s - 1$$
$$|\mathcal{V}(S_{n+1,s})| = |U_{n+1,s}| + |\{\mathbf{0}\}| = (s+1)^{n+1}$$

A family of convex polytopes: The "bottom-left quadrant"

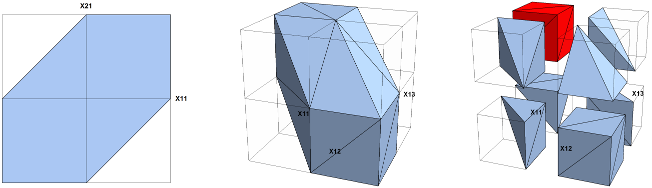

Figure: $P_{2,1}$ and $P_{3,1}$

Left to show: Normed volume of $P_{n,s}$ = Normed volume of $S_{n+1,s}$

Counting vertices:
$$|U_{n,s}| = (s+1)^n - 1$$
$$|V_{n,s}| = s((s+1)^n - 1)$$

$$|\mathcal{V}(P_{n,s})| = |U_{n,s}| + |V_{n,s}| = (s+1)^{n+1} - s - 1$$
$$|\mathcal{V}(S_{n+1,s})| = |U_{n+1,s}| + |\{\mathbf{0}\}| = (s+1)^{n+1}$$

$$|\mathcal{V}(S_{n+1,s})| - |\mathcal{V}(P_{n,s})| = s + 1$$

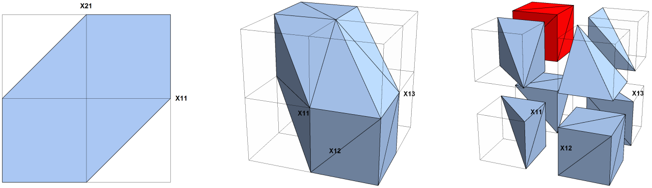

Figure: $P_{2,1}$ and $P_{3,1}$

Left to show: Normed volume of $P_{n,s}$ = Normed volume of $S_{n+1,s}$

Theorem

Let

$$M_{n,s} := \begin{pmatrix} & -I_s \\ I_{ns} & \vdots \\ & -I_s \end{pmatrix} = \begin{pmatrix} 1 & & 0 & -1 & & 0 \\ & \ddots & & 0 & \ddots & -1 \\ & & \ddots & & \vdots & \\ & & & -1 & \ddots & 0 \\ 0 & & 1 & 0 & & -1 \end{pmatrix} \in \mathbb{R}^{(ns)\times((n+1)s)}$$

Then $M_{n,s}S_{n+1,s} = P_{n,s}$

A family of convex polytopes: A linear transformation

$$M_{n,s} := \begin{pmatrix} & -I_s \\ I_{ns} & \vdots \\ & -I_s \end{pmatrix} = \begin{pmatrix} 1 & & 0 & -1 & \ddots & 0 \\ & \ddots & & 0 & \vdots & -1 \\ & & \ddots & -1 & & 0 \\ 0 & & 1 & 0 & \ddots & -1 \end{pmatrix} \in \mathbb{R}^{(ns)\times((n+1)s)}$$

$M_{n,s}S_{n+1,s} = P_{n,s}$

What happens to the vertices?

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A family of convex polytopes: A linear transformation

$$M_{n,s} := \begin{pmatrix} & -I_s \\ I_{ns} & \vdots \\ & -I_s \end{pmatrix} = \begin{pmatrix} 1 & & 0 & -1 & & 0 \\ & \ddots & & 0 & \ddots & -1 \\ & & \ddots & -1 & \vdots & 0 \\ 0 & & 1 & 0 & \ddots & -1 \end{pmatrix} \in \mathbb{R}^{(ns)\times((n+1)s)}$$

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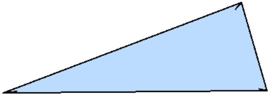
Problem: Everything can happen under projection!

A family of convex polytopes: A linear transformation



```
Vertex count: 4
Volume: 5.33333
```

Remember: $P_{n,s}$ is the "shadow" of $M'_{n,s} S_{n+1,s}$

```
Vertex count: 4
Volume: 9.
```

A family of convex polytopes: A linear transformation

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A family of convex polytopes: A special triangulation

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A family of convex polytopes: A special triangulation

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A family of convex polytopes: A special triangulation

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- Normed volumes of projections: $1, 1, 2, 1, 1, 2, 2, 2, 3, 4,$
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A family of convex polytopes: A special triangulation

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A family of convex polytopes: A special triangulation

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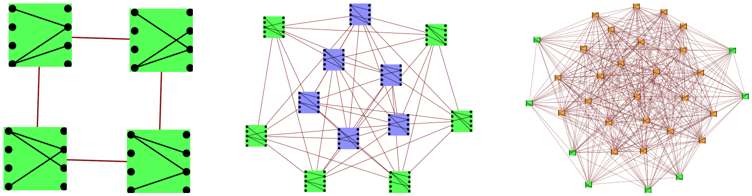

A family of convex polytopes: A special triangulation

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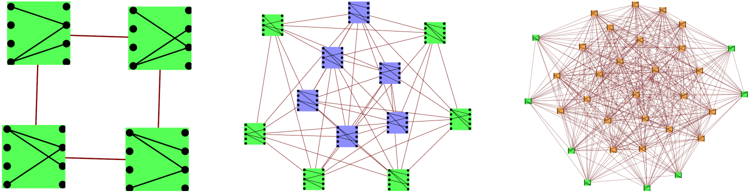

Indirect approach: Prove existence of a suitable decomposition

Lifting theorem (Kerber, Tichy, W.)

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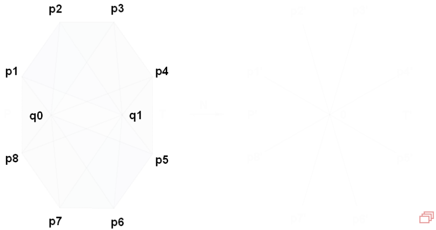

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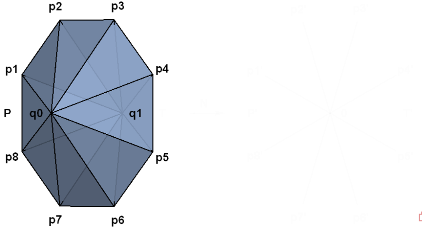

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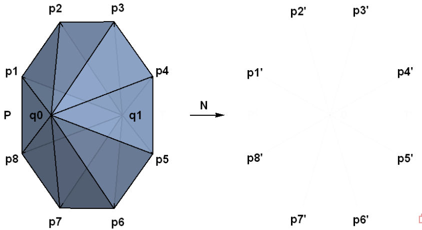

A family of convex polytopes: A special triangulation

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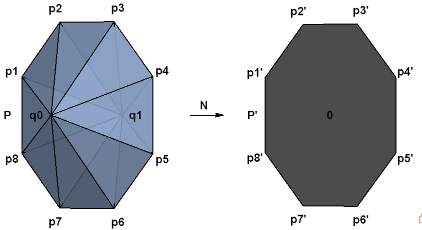

A family of convex polytopes: A special triangulation

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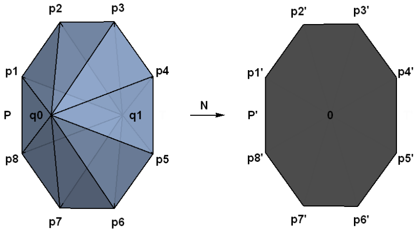

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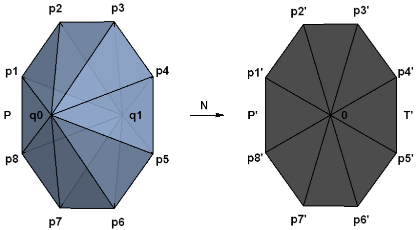
- $\mathcal{T}'$ a triangulation of $\{\mathbf{p}'_1, \dots, \mathbf{p}'_m, \mathbf{0}\}$ such that
  - $\mathbf{0}$ is a common vertex of all simplices in $\mathcal{T}'$



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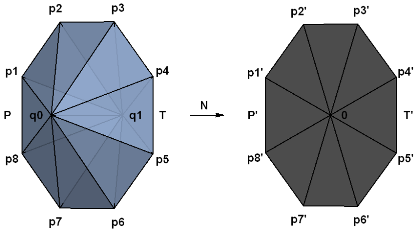
  - $\mathbf{p}'_1, \ldots, \mathbf{p}'_m$ are in convex position, $\mathbf{p}'_i := N\mathbf{p}_i$
  - $N\mathbf{q}_1 = \cdots = N\mathbf{q}_n = \mathbf{0} \in \operatorname{int}(P')$, $P' := \operatorname{conv}(\{\mathbf{p}'_1, \ldots, \mathbf{p}'_m\})$
  - $P \subseteq \bigcup \{\operatorname{conv}(\{\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_{d'}}, \mathbf{q}_1, \ldots, \mathbf{q}_n\}) \mid i_1, \ldots, i_{d'} \in \{1, \ldots, m\}\}$
- $\mathcal{T}'$ a triangulation of $\{\mathbf{p}'_1, \ldots, \mathbf{p}'_m, \mathbf{0}\}$ such that
  - $\mathbf{0}$ is a common vertex of all simplices in $\mathcal{T}'$
- $\mathcal{T} := \{\operatorname{conv}(\{\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_{d'}}, \mathbf{q}_1, \ldots, \mathbf{q}_n\}) \mid \operatorname{conv}(\{\mathbf{p}'_{i_1}, \ldots, \mathbf{p}'_{i_{d'}}, \mathbf{0}\}) \in \mathcal{T}'\}$

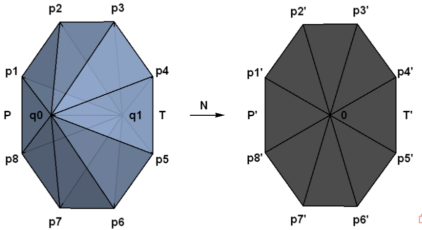


A family of convex polytopes: A special triangulation

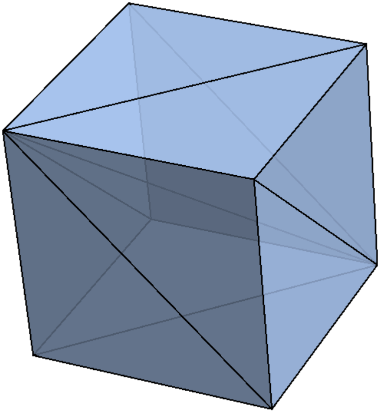
Lifting theorem (Kerber, Tichy, W.)

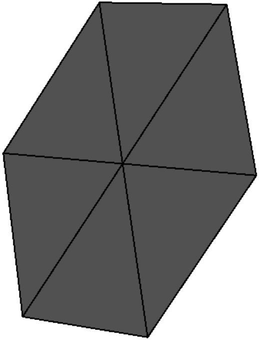
Let
- $d, d' \in \mathbb{N}$, $d' < d$, $N := (I_{d'}, 0) \in \mathbb{R}^{d' \times d}$ (projection matrix)
- $m \in \mathbb{N}$, $n := d - d' + 1$
- $\mathbf{p}_1, \ldots, \mathbf{p}_m, \mathbf{q}_1, \ldots, \mathbf{q}_n \in \mathbb{R}^d$ in convex position such that
  - $P := \operatorname{conv}(\{\mathbf{p}_1, \ldots, \mathbf{p}_m, \mathbf{q}_1, \ldots, \mathbf{q}_n\})$ is non-degenerate
  - $\mathbf{p}'_1, \ldots, \mathbf{p}'_m$ are in convex position, $\mathbf{p}'_i := N\mathbf{p}_i$
  - $N\mathbf{q}_1 = \cdots = N\mathbf{q}_n = \mathbf{0} \in \operatorname{int}(P')$, $P' := \operatorname{conv}(\{\mathbf{p}'_1, \ldots, \mathbf{p}'_m\})$
  - $P \subseteq \bigcup\{\operatorname{conv}(\{\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_{d'}}, \mathbf{q}_1, \ldots, \mathbf{q}_n\}) \mid i_1, \ldots, i_{d'} \in \{1, \ldots, m\}\}$
- $\mathcal{T}'$ a triangulation of $\{\mathbf{p}'_1, \ldots, \mathbf{p}'_m, \mathbf{0}\}$ such that
  - $\mathbf{0}$ is a common vertex of all simplices in $\mathcal{T}'$
- $\mathcal{T} := \{\operatorname{conv}(\{\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_{d'}}, \mathbf{q}_1, \ldots, \mathbf{q}_n\}) \mid \operatorname{conv}(\{\mathbf{p}'_{i_1}, \ldots, \mathbf{p}'_{i_{d'}}, \mathbf{0}\}) \in \mathcal{T}'\}$

Then $\mathcal{T}$ is a triangulation of $\{\mathbf{p}_1, \ldots, \mathbf{p}_m, \mathbf{q}_1, \ldots, \mathbf{q}_n\}$

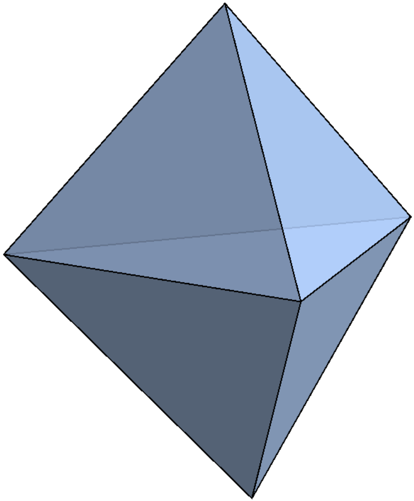


A family of convex polytopes: A special triangulation





A family of convex polytopes: A special triangulation



Theorem

$P_{n,s}$ and $S_{n+1,s}$ satisfy the assumptions of the lifting theorem

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Theorem

In the case of $P_{n,s}$ and $S_{n+1,s}$ the normed volumes of corresponding simplices are equal

Thank you for your attention!