Shift Radix Systems - some new characterization results and topological properties

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$\mathcal{D}_d^{(0)} \subseteq \mathcal{D}_d$

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Orbit of $(0,3)$ ultimately periodic!
$\mathbf{r} \in \mathcal{D}_2$?

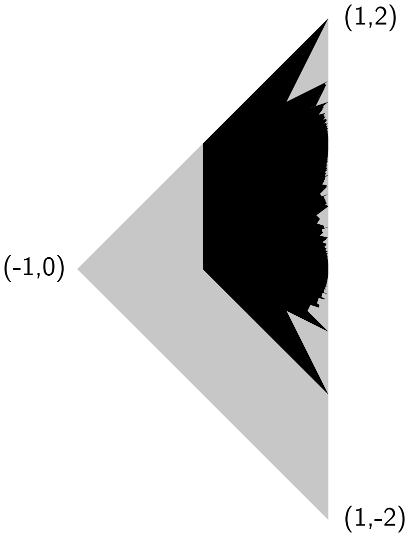

Figure: $\mathcal{D}_2^{(0)}$ in $\mathcal{D}_2$

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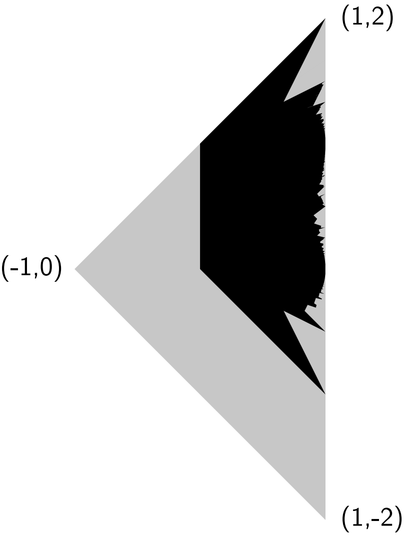

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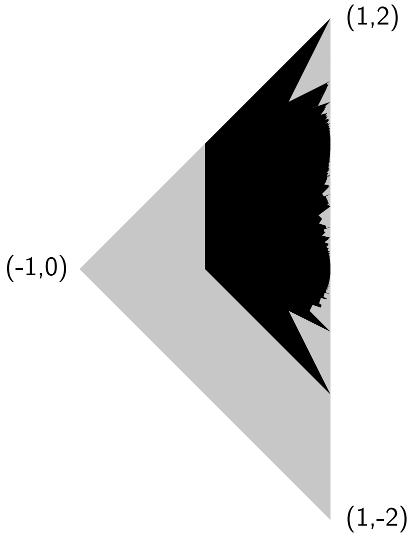

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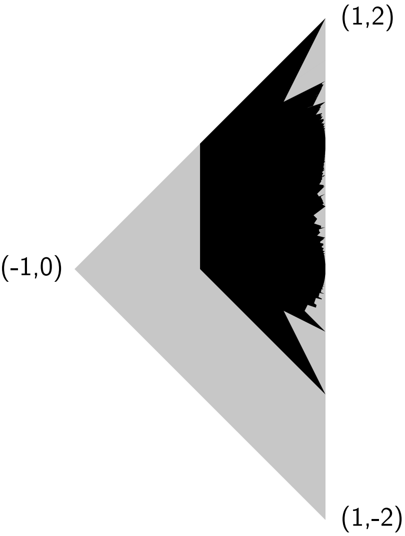
Relation between SRS and β-Expansions



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Relation between SRS and Canonical Number Systems

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Then $\mathcal{A} := \{0, 1, \ldots, \lfloor \beta \rfloor\}$ is called the set of digits,
as every $\gamma \in [0, \infty)$ can be represented uniquely in the form

$$\gamma = a_m \beta^m + a_{m-1} \beta^{m-1} + \cdots$$

(greedy expansion of γ with respect to β)

with $m \in \mathbb{Z}$ and $a_i \in \mathcal{A}$, such that

$$0 \leq \gamma - \sum_{i=k}^{m} a_i \beta^i < \beta^k$$

holds for all $k \leq m$.

Motivation - Relation to β-Expansions

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In that case β is an algebraic integer (furthermore a Pisot number) and therefore has a minimal polynomial

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Then β has property (F) $\iff$ $(r_0, \ldots, r_{d-2}) \in \mathcal{D}_{d-1}^{(0)}$ (AKIYAMA et al. 2005)

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For $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$ let

$$R_{\mathbf{r}} := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -r_1 & -r_2 & \cdots & \cdots & -r_d \end{pmatrix}$$

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- $\partial\mathcal{D}_d = \{\mathbf{r} \in \mathbb{R}^d \mid \rho(R_{\mathbf{r}}) = 1\}$

Characterization of $\mathcal{D}_d^{(0)}$ – Two important concepts

Cutout polyhedra

For a tuple π of vectors in $\mathbb{Z}^d$ let $\mathcal{P}(\pi)$ denote the set of all $\mathbf{r} \in \mathbb{R}^d$ for which π is a period of $\tau_{\mathbf{r}}$.

$\pi = (\mathbf{x_1}, \dots, \mathbf{x_n})$, $\tau_{\mathbf{r}}(\mathbf{x_1}) = \mathbf{x_2}$, $\dots$,$\tau_{\mathbf{r}}(\mathbf{x_n}) = \mathbf{x_1}$

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$\mathcal{D}_d^{(0)} = \mathcal{D}_d \setminus \bigcup_{\pi \neq 0} \mathcal{P}(\pi)$ (BRUNOTTE 2001)

Sets of witnesses

A set $V \subseteq \mathbb{Z}^d$ is called a set of witnesses for $\mathbf{r} \in \mathbb{R}^d$ iff it is stable under $\tau_{\mathbf{r}}$ and $\tau_{\mathbf{r}}^{\star} := -\tau_{\mathbf{r}} \circ (-\mathrm{id}_{\mathbb{Z}^d})$ and contains a generating set of the group $(\mathbb{Z}^d, +)$ which is closed under taking inverses.

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Every such set of witnesses has the decisive property:

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Find a finite set of witnesses iteratively for $\mathbf{r} \in \mathrm{int}\,(\mathcal{D}_d)$:

$$\begin{aligned} V_0 &:= \{(\pm 1, 0, \dots, 0), \dots, (0, \dots, 0, \pm 1)\} \\ \forall\, n \in \mathbb{N} : V_n &:= V_{n-1} \cup \tau_{\mathbf{r}}(V_{n-1}) \cup \tau_{\mathbf{r}}^{\star}(V_{n-1}) \\ V_{\mathbf{r}} &:= \bigcup_{n \in \mathbb{N}_0} V_n \end{aligned}$$

(BRUNOTTE 2001)

Characterization of $\mathcal{D}_d^{(0)}$ – Previous results

- $\mathcal{D}_1 = [-1, 1]$, $\mathcal{D}_1^{(0)} = [0, 1)$

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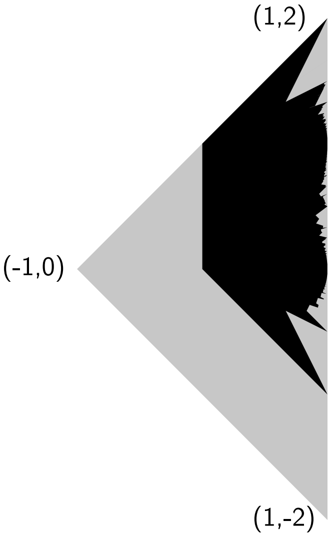

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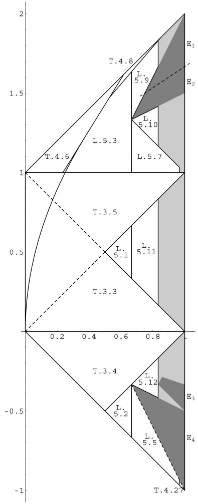

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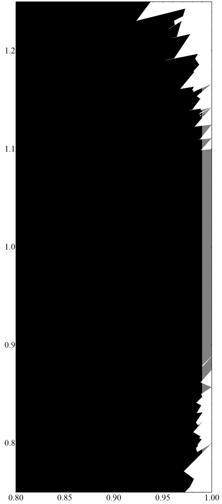
  Applied by SURER 2008 to characterize $\mathcal{D}_2^{(0)} \cap \left\{(x, y) \in \mathbb{R}^2 \mid x \leq L\right\}$ where $L = \frac{99}{100}$



Characterization of $\mathcal{D}_d^{(0)}$ – New results

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- $\mathcal{D}_2^{(0)}$ has at least 22 connected components
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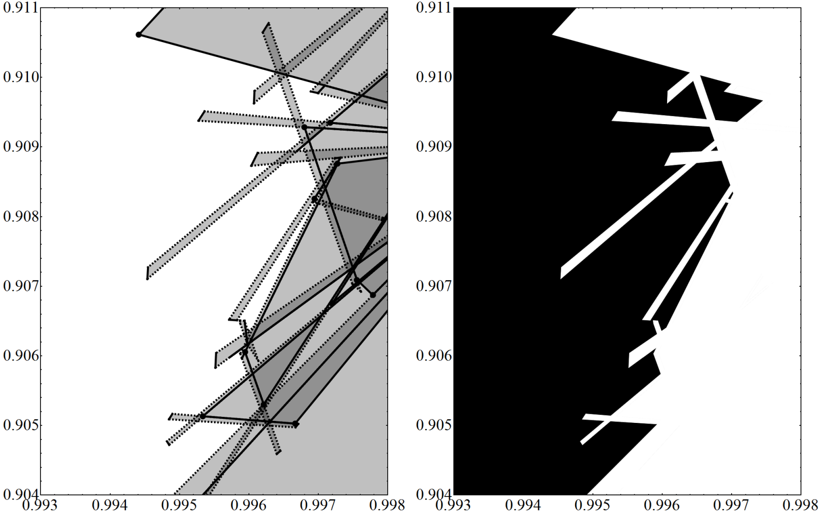
Result achieved by a new algorithm which has been used to characterize $\mathcal{D}_2^{(0)} \cap \left\{(x, y) \in \mathbb{R}^2 \mid x \leq L\right\}$ where $L = \frac{511}{512}$.

Characterization of $\mathcal{D}_d^{(0)}$ – New results



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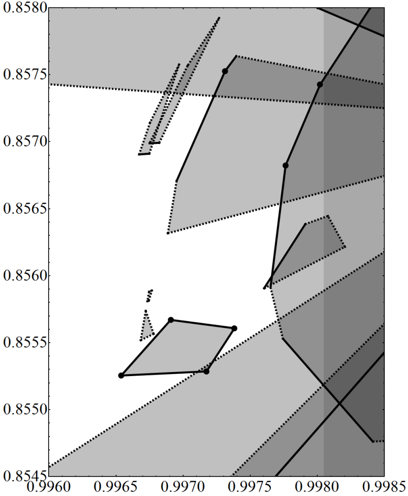

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Handle classes in a sophisticated order to minimize computation time!

Advantages of the first algorithm:

- Faster than Brunotte's algorithm

Two new algorithms

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- A "real" algorithm (terminates for all inputs)

Advantages of the second algorithm:

- Much faster than Brunotte's algorithm
- Very compact output (minimal list of cutout polyhedra)

Thank you for your attention!

Let
$P(X) = X^d + p_{d-1}X^{d-1} + \cdots + p_1 X + p_0 \in \mathbb{Z}[X]$
$\mathcal{R} := \mathbb{Z}[X]/P(X)\mathbb{Z}[X]$
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$(P, \mathcal{N})$ is called a CNS, P a CNS polynomial and $\mathcal{N}$ the set of digits if every non-zero element $A(x) \in \mathcal{R}$ can be represented uniquely in the form

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Motivation - Relation to Canonical Number Systems

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Then P is a CNS polynomial $\iff (\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \ldots, \frac{p_2}{p_0}, \frac{p_1}{p_0}) \in \mathcal{D}_d^{(0)}$