Gaussian Shift Radix Systems (GSRS)
Pethő's Loudspeaker

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Generalized Radix Representations and Dynamical Systems I

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Shift Radix Systems for Gaussian Integers and Pethő's Loudspeaker

H. BRUNOTTE, P. KIRSCHENHOFER,
J. M. THUSWALDNER

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is called the d - *dimensional SRS associated with* $\mathbf{r}$

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Analogously for $d \in \mathbb{N}$ and $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{C}^d$

$$\begin{aligned} \gamma_{\mathbf{r}} : \mathbb{Z}[\mathrm{i}]^d &\mapsto \mathbb{Z}[\mathrm{i}]^d \\ \mathbf{x} = (x_1, \dots, x_d) &\to (x_2, \dots, x_d, -\lfloor \mathbf{rx} \rfloor) \end{aligned}$$

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where $\lfloor y \rfloor := \lfloor \Re y \rfloor + \mathrm{i} \lfloor \Im y \rfloor$ for some complex y. (complex floor)

For $d \in \mathbb{N}$

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Orbit of 2 ultimately periodic!
$r \in \mathcal{G}_1$?

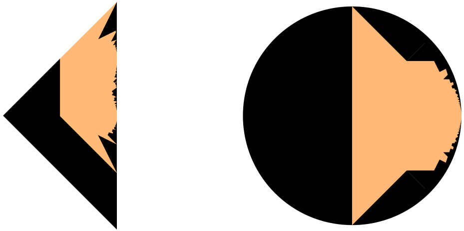

Figure: $\mathcal{D}_2^{(0)}$ in $\mathcal{D}_2$ and $\mathcal{G}_1^{(0)}$ in $\mathcal{G}_1$

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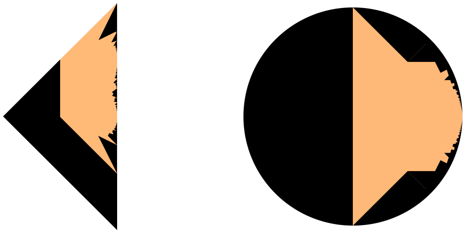

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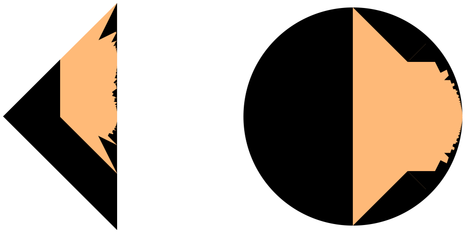
Motivation



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Relation between SRS, β-Expansions and Canonical Number Systems

Motivation - Relation to β-Expansions

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Then $\mathcal{A} := \{0, 1, \ldots, \lfloor \beta \rfloor\}$ is called the set of digits,
as every $\gamma \in [0, \infty)$ can be represented uniquely in the form

$\gamma = a_m \beta^m + a_{m-1} \beta^{m-1} + \cdots$
(greedy expansion of γ with respect to β)

with $m \in \mathbb{Z}$ and $a_i \in \mathcal{A}$, such that

$$0 \leq \gamma - \sum_{i=k}^{m} a_i \beta^i < \beta^k$$

holds for all $k \leq m$.

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In that case β is an algebraic integer (furthermore a Pisot number) and therefore has a minimal polynomial

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Then β has property (F) $\iff$ $(r_0, \ldots, r_{d-2}) \in \mathcal{D}_{d-1}^{(0)}$

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Let $P(X) = X^d + p_{d-1}X^{d-1} + \cdots + p_1 X + p_0 \in \mathbb{Z}[X]$

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Then P is a CNS polynomial $\iff$ $(\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \ldots, \frac{p_2}{p_0}, \frac{p_1}{p_0}) \in \mathcal{D}_d^{(0)}$

For $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$ let

$$R_{\mathbf{r}} := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -r_1 & -r_2 & \cdots & \cdots & -r_d \end{pmatrix}$$

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Equivalent statements are true for $\mathcal{G}_d$.

For a tuple π of vectors in $\mathbb{Z}^d$ let $\mathcal{P}(\pi)$ denote the set of all $\mathbf{r} \in \mathbb{R}^d$ for which π is a period of $\tau_{\mathbf{r}}$.

$\pi = (\mathbf{x_1}, \ldots, \mathbf{x_n})$, $\tau_{\mathbf{r}}(\mathbf{x_1}) = \mathbf{x_2}$, $\ldots$,$\tau_{\mathbf{r}}(\mathbf{x_n}) = \mathbf{x_1}$

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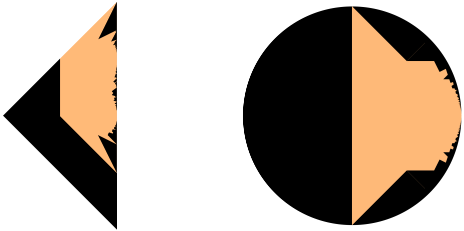

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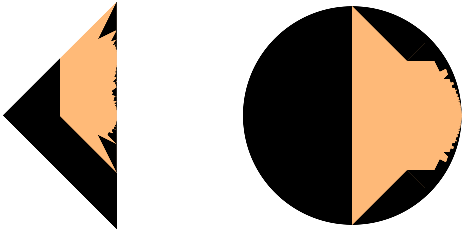

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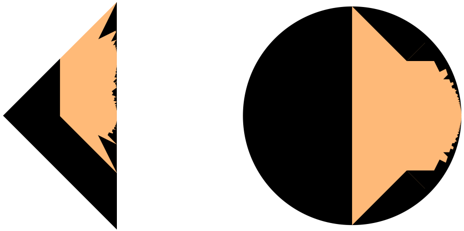

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Hope that characterization of $\mathcal{G}_1^{(0)}$ is of an intermediate level of difficulty.

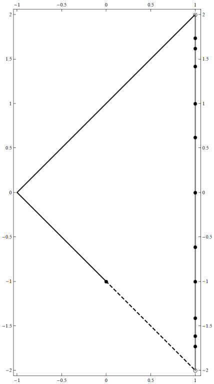

Figure: $\mathcal{D}_2$

Points on right line: $\frac{\pm 1 \pm \sqrt{5}}{2}$, $\pm\sqrt{2}$, $\pm\sqrt{3}$ (quadratic irrational numbers)

$$\{r \in \mathbb{C} \mid |r| < 1\} \subseteq \mathcal{G}_1 \subseteq \{r \in \mathbb{C} \mid |r| \leq 1\}$$

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An easy argument for the first relation:

$|\gamma_r(x)| < |r||x| + \sqrt{2} \Rightarrow$

$\Rightarrow$ If $|r| < 1$ then $r \in \mathcal{G}_1$

All cycles of r are contained in
$\{x \in \mathbb{Z}[\mathrm{i}] \mid |x| < \frac{\sqrt{2}}{1-|r|} + \sqrt{2}\}$

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Methods carry over to SRS and GSRS of arbitrary dimension.

Indirect argument:

Recapitulation:

$r \in \mathbb{C}$

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$n \in \mathbb{Z}$, $x \in \mathbb{R}$

$n = \lfloor x \rfloor \iff 0 \leq x - n < 1$

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$n \in \mathbb{Z}$, $x \in \mathbb{R}$

$n = \lfloor x \rfloor \iff 0 \le x - n < 1$

$x + \mathrm{i}y \in \mathbb{C}$, $a + \mathrm{i}b \in \mathbb{Z}[\mathrm{i}]$, $A + \mathrm{i}B \in \mathbb{Z}[\mathrm{i}]$

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If π has n entries then $\mathcal{P}(\pi)$ is given by $4n$ integer linear inequalities

Example:

Let $\pi = (1)$

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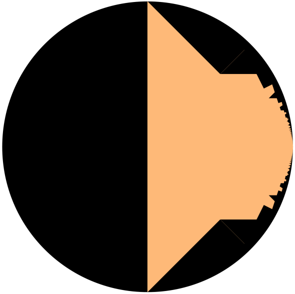

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$\mathcal{G}_1^{(0)}$ has empty intersection with the second quadrant.

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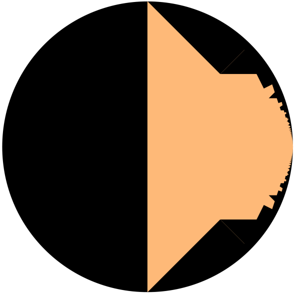

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As $(x_1, \ldots, x_n)$ is a cycle of $r \iff (\mathrm{i}\overline{x_1}, \ldots, \mathrm{i}\overline{x_n})$ is a cycle of $\overline{r}$

$\mathcal{G}_1^{(0)}$ is symmetric and therefore all of its elements have real parts ≥ 0.

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Theorem:
Let $r \in \mathbb{C}$ and $\mathcal{Z}_r$ be the set of all elements of $\mathbb{Z}[\mathrm{i}]$, whose orbits end up in 0.

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Then $r \in \mathcal{G}_1^{(0)}$

Find V recursively: $V_1 := \{1, -1, \mathrm{i}, -\mathrm{i}\}$
$V_n := V_{n-1} \cup \{S_i\gamma_r(x) \mid x \in V_{n-1} \wedge i \in \{1, \ldots, 4\}\}$
$V := \cup_{n=1}^{\infty} V_n$

$S_1 f(x) = f(x)$
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The complex case - Pethő's Loudspeaker

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Idea leads to an algorithm by Brunotte, which always calculates $\mathcal{G}_1^{(0)} \cap conv(r_1, \ldots, r_n)$ i.e. the intersection of $\mathcal{G}_1^{(0)}$ with the convex hull of finitely many interior points of $\mathcal{G}_1$ in finitely many steps.

Example: V for $r = \frac{9}{10} + \mathrm{i}\,\frac{6}{17}$

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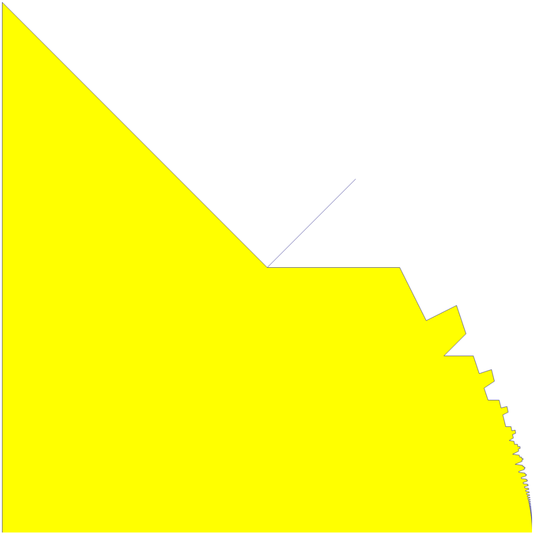
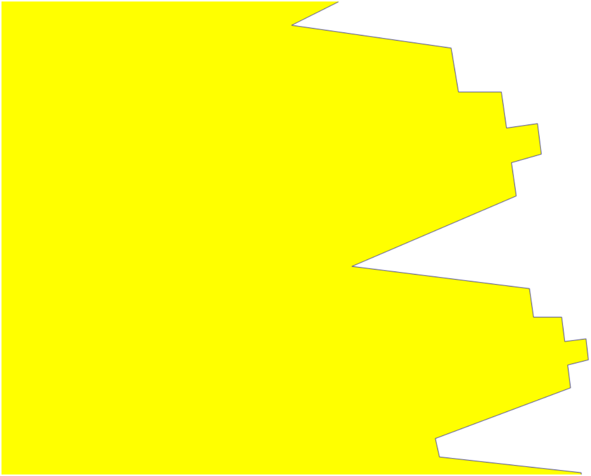
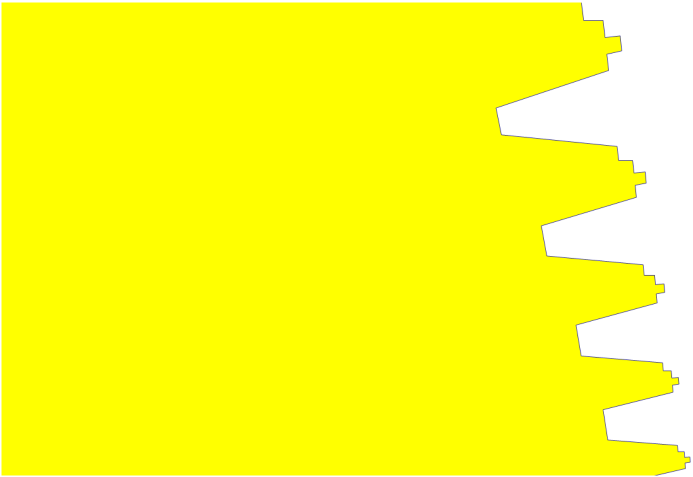
For further considerations let $\mathcal{V}$ denote the set of the $4 \cdot |V|$ arrows on V (images under $S_1 f, \ldots, S_4 f$).







10 Points:

$$z_1(n) = 1 + \tfrac{-2+\mathrm{i}n}{n^2-2}$$

$$z_2(n) = 1 + \tfrac{-1+\mathrm{i}(n-1)}{n^2-n-1}$$

$$z_3(n) = 1 + \tfrac{-1+\mathrm{i}(n-1)}{n^2-n}$$

$$z_4(n) = 1 + \tfrac{-1+\mathrm{i}n}{n^2}$$

$$z_5(n) = 1 + \tfrac{-1+\mathrm{i}n}{n^2+1}$$

$$z_6(n) = 1 + \tfrac{-1+\mathrm{i}(n+1)}{n^2+n+1}$$

$$z_7(n) = 1 + \tfrac{-1+\mathrm{i}(n+1)}{n^2+n+2}$$

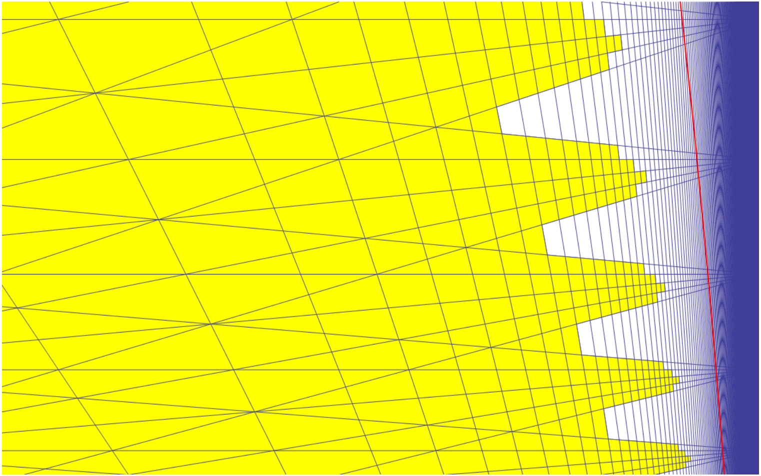
$$z_8(n) = 1 + \tfrac{-1+\mathrm{i}n}{n^2+2}$$

$$z_9(n) = 1 + \tfrac{-1+\mathrm{i}n}{n^2+3}$$

$$z_{10}(n) = 1 + \tfrac{-2+\mathrm{i}(n+1)}{n^2+n+6}$$

Is $\mathcal{G}_1^{(0)}$ star-shaped?

Some observations and results



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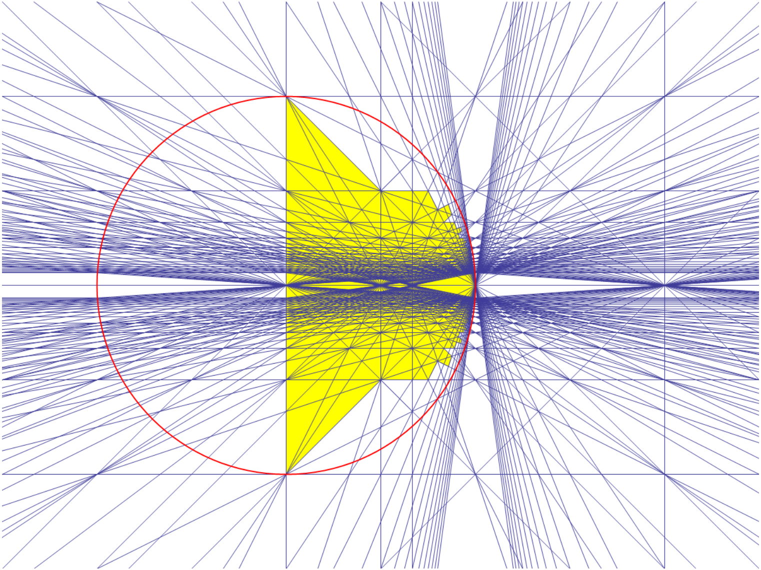


6 families of lines:

$$(0,0)+t(p,1)$$

$$(\tfrac{1}{2},0)+t(p,2)$$

$$(\tfrac{2}{3},0)+t(p,3)$$

$$(1,0)+t(-2,p)$$

$$(2,0)+t(p,-1)$$

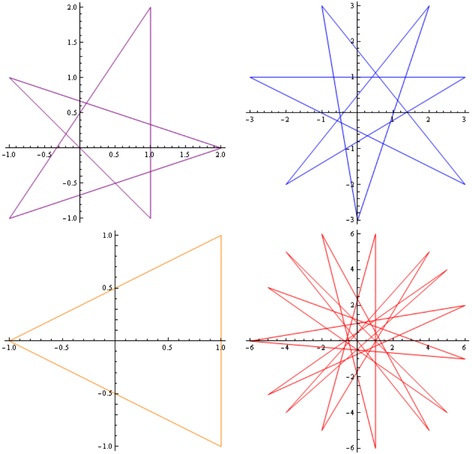
$$(0,\tfrac{1}{p})+t(1,0)$$

The corresponding polygons of which cycles "touch"

$$\mathcal{G}_1^{(0)} = \mathcal{G}_1 \setminus \bigcup_{\pi \neq 0} \mathcal{P}(\pi)$$

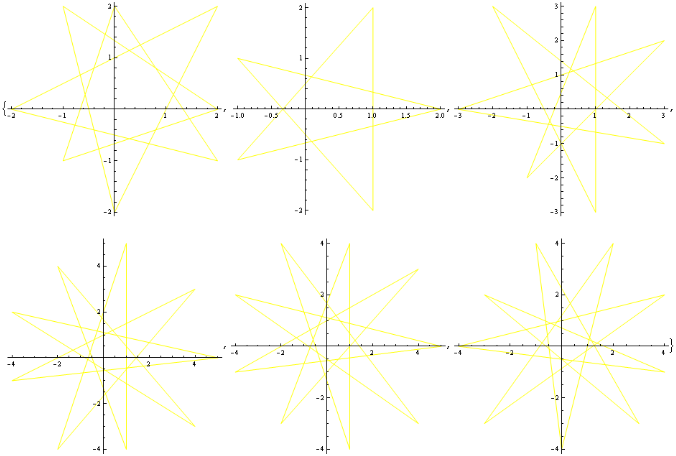
?

Cut outs: 4 families of cycles



Cut outs: 4 families of cycles

Cut outs: 6 additional cycles



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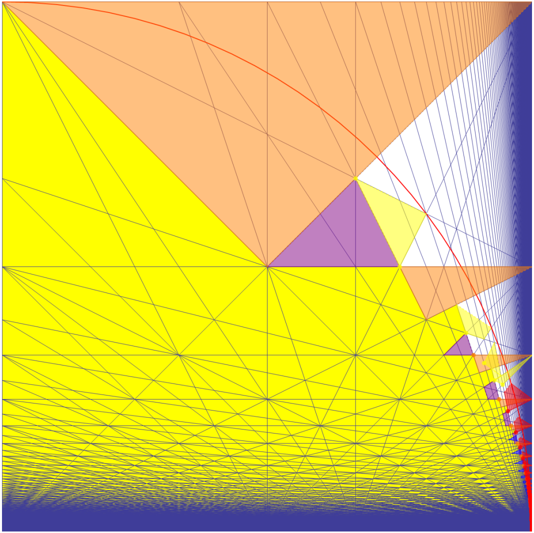

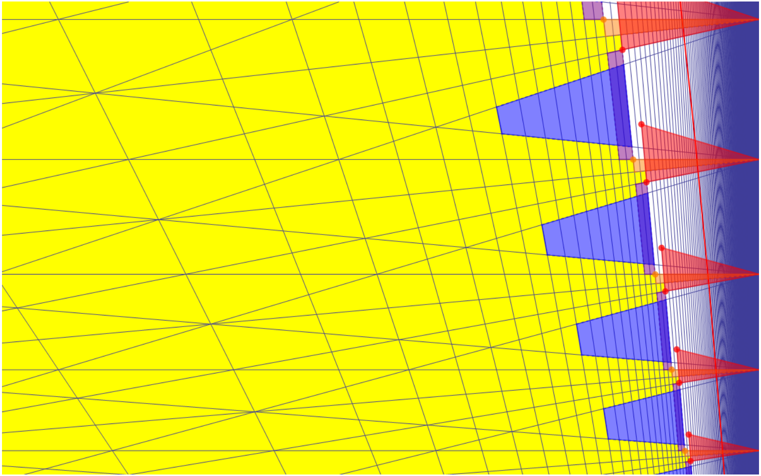

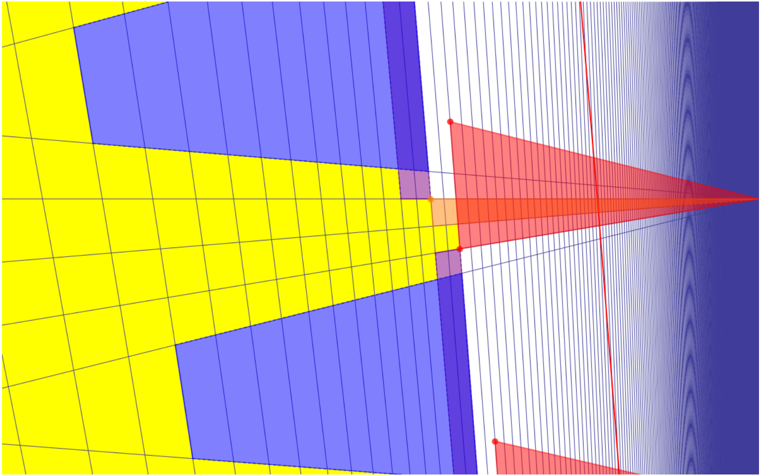

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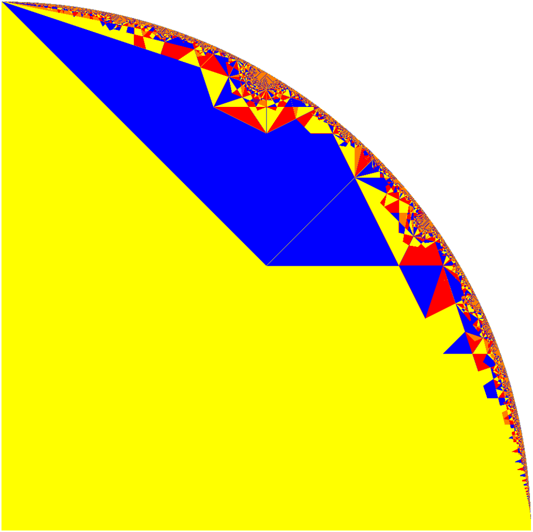
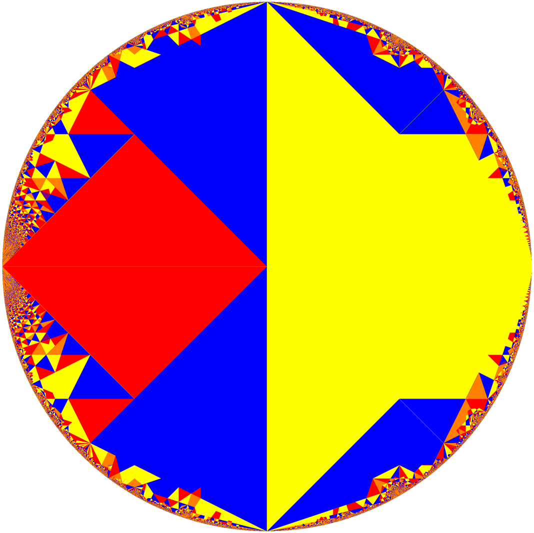
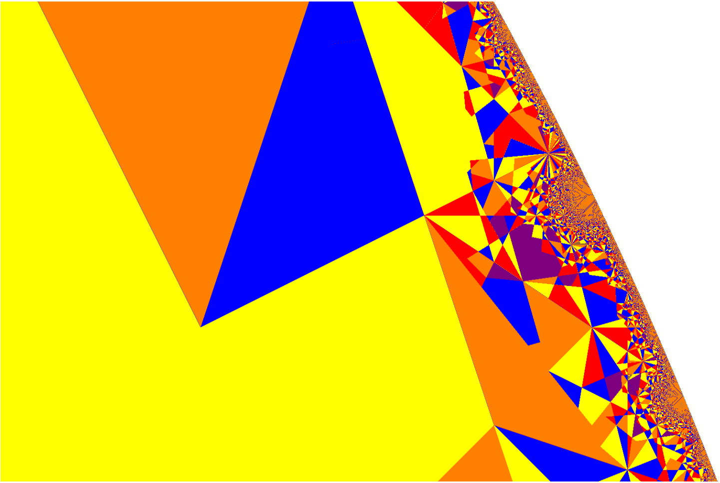
What is the "cycle structure" of $\mathcal{G}_1$?







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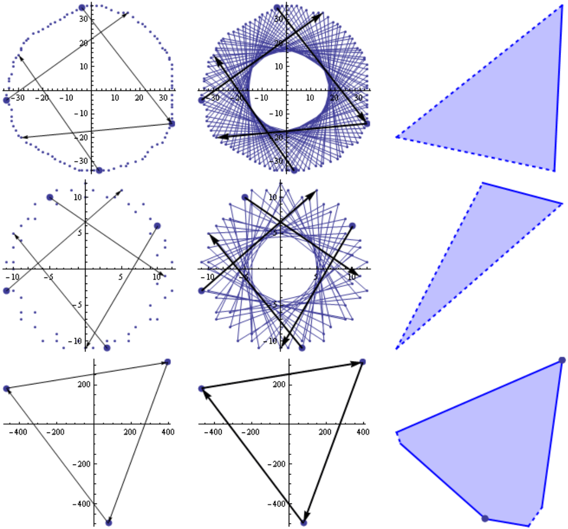
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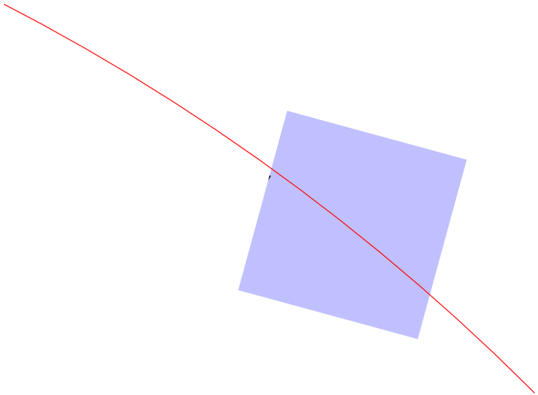
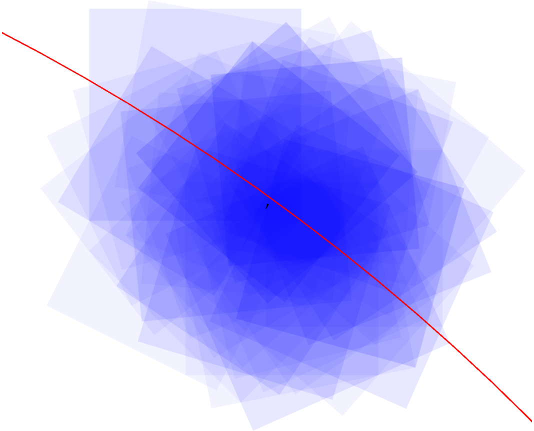
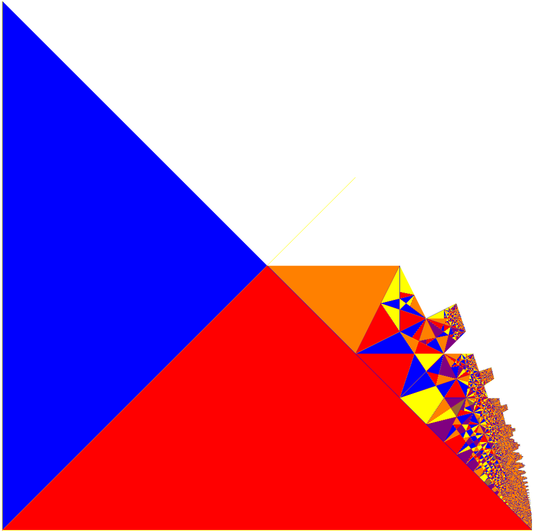


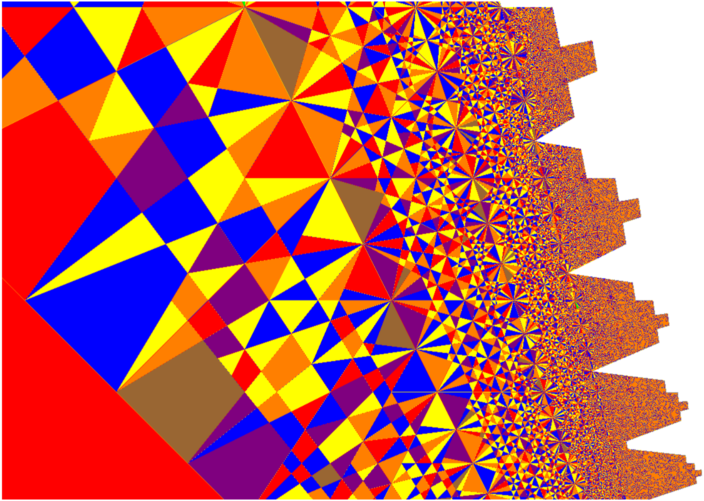


A possible direct proof of the loudspeaker's structure:

Calculating the polygon $\pi(\mathcal{V})$ for a given $\mathcal{V}$
(set of arrows on V).

Some observations and results





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Therefore every subset of the interior of the unit circle intersects with only finitely many polygons $\pi(\mathcal{V})$.

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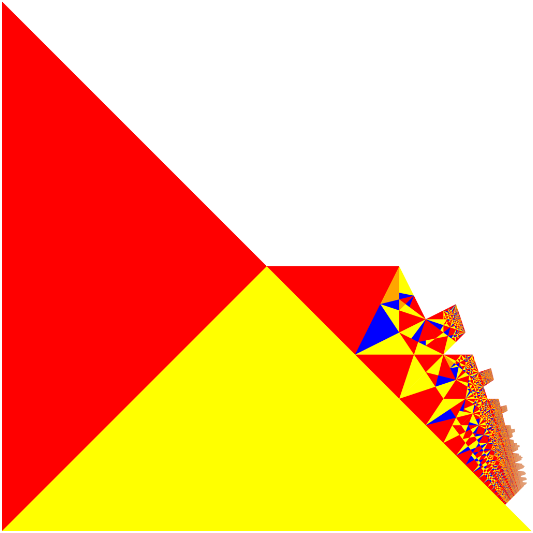
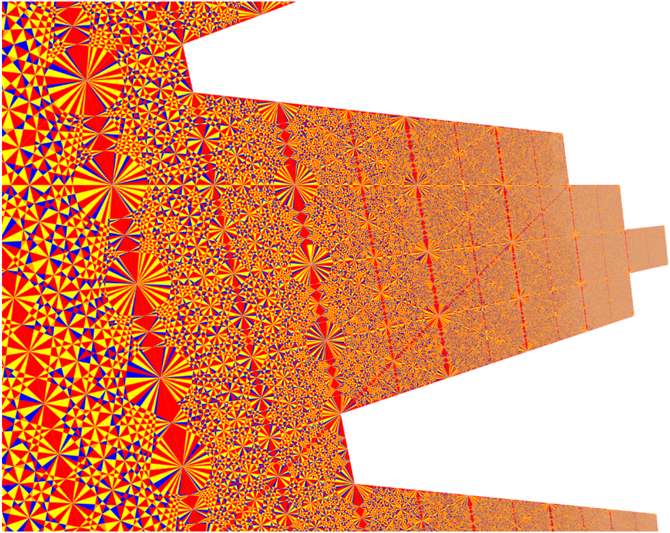
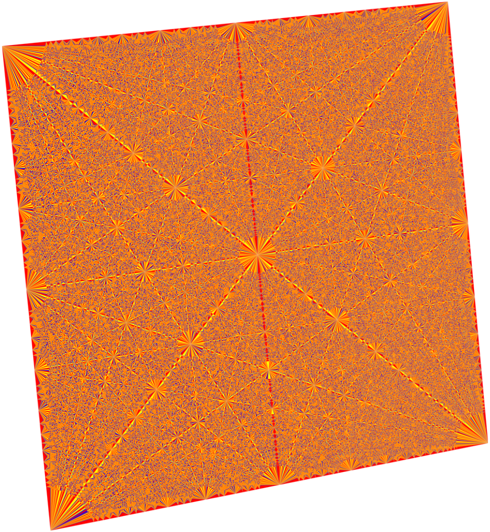
More detailed images, understanding of the behavior close to 1, recognition of possible self-similarities...







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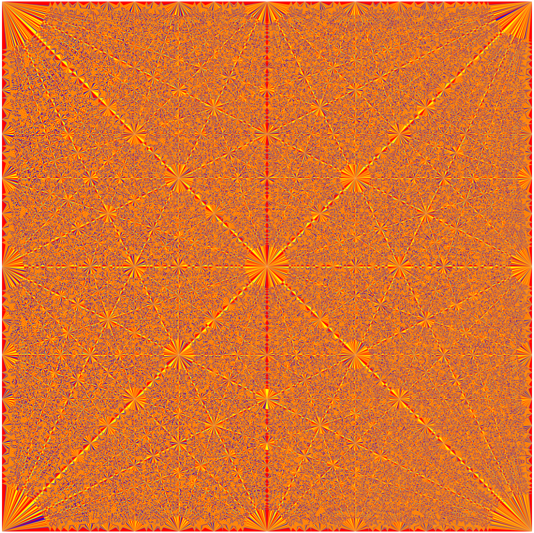

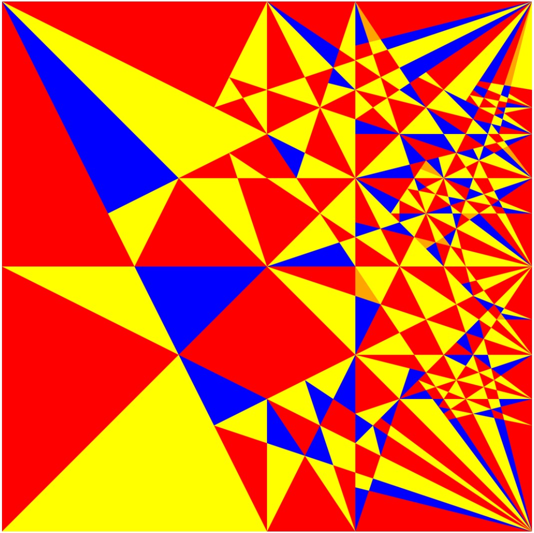

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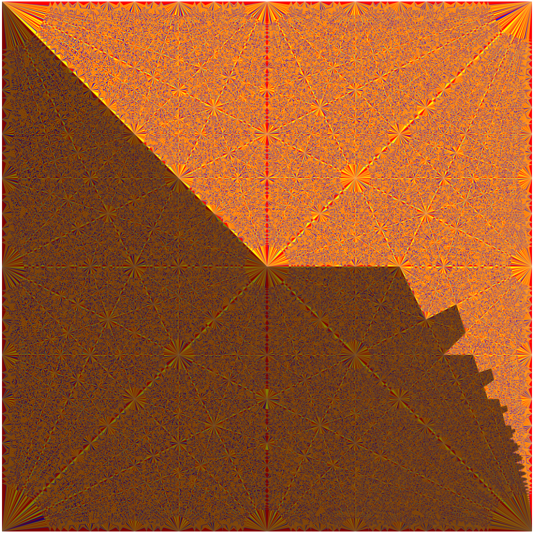

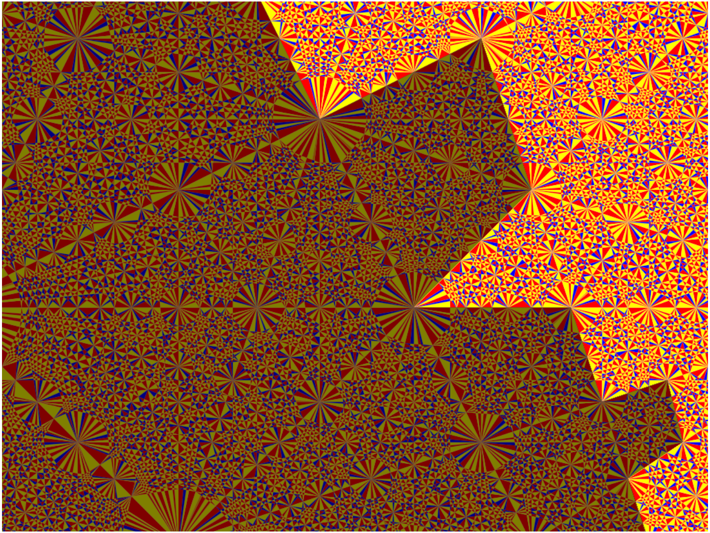

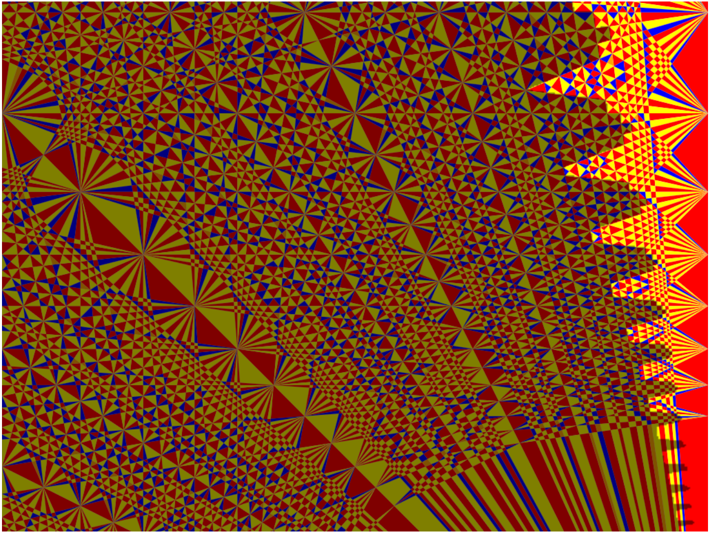

Thank you for your attention!

Motivation - Relation to Canonical Number Systems

Let
$P(X) = X^d + p_{d-1}X^{d-1} + \cdots + p_1 X + p_0 \in \mathbb{Z}[X]$
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$(P, \mathcal{N})$ is called a CNS, P a CNS polynomial and $\mathcal{N}$ the set of digits if every non-zero element $A(x) \in \mathcal{R}$ can be represented uniquely in the form

$$A(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0$$

with $m \in \mathbb{N}_0$, $a_i \in \mathcal{N}$ and $a_m \neq 0$.

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Then P is a CNS polynomial $\iff$ $(\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \ldots, \frac{p_2}{p_0}, \frac{p_1}{p_0}) \in \mathcal{D}_d^{(0)}$