Gaussian Shift Radix Systems (GSRS)
Pethő's Loudspeaker

Mario Weitzer

March 29, 2012

Generalized Radix Representations and Dynamical Systems I

S. AKIYAMA, T. BORBÉLY, H. BRUNOTTE,
A. PETHŐ, J. M. THUSWALDNER

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Shift Radix Systems for Gaussian Integers and Pethő's Loudspeaker

H. BRUNOTTE, P. KIRSCHENHOFER,
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June 14, 2011. (to appear)

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$$\tau_{\mathbf{r}} : \mathbb{Z}^d \mapsto \mathbb{Z}^d$$
$$\mathbf{x} = (x_1, \dots, x_d) \to (x_2, \dots, x_d, -\lfloor \mathbf{rx} \rfloor)$$

is called the d - *dimensional SRS associated with* $\mathbf{r}$

where $\mathbf{rx} = \sum_{i=1}^{d} r_i x_i$ denotes the scalar product of $\mathbf{r}$ and $\mathbf{x}$
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Analogously for $d \in \mathbb{N}$ and $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{C}^d$

$$\begin{aligned} \gamma_{\mathbf{r}} : \mathbb{Z}[\mathrm{i}]^d &\mapsto \mathbb{Z}[\mathrm{i}]^d \\ \mathbf{x} = (x_1, \dots, x_d) &\rightarrow (x_2, \dots, x_d, -\lfloor \mathbf{r}\mathbf{x} \rfloor) \end{aligned}$$

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where $\lfloor y \rfloor := \lfloor \Re y \rfloor + \mathrm{i} \lfloor \Im y \rfloor$ for some complex y. (complex floor)

Example:

$d = 4$
$\mathbf{r} = (-\frac{3}{10} + \frac{5}{4}\mathrm{i}, -\frac{\pi^2}{6}, -\frac{2}{9} + \frac{1}{7}\mathrm{i}, 17 - \mathrm{i}) \in \mathbb{C}^4$
$\mathbf{x} = (-4 + \mathrm{i}, 3 - 2\mathrm{i}, -12 - 5\mathrm{i}, -7 + 2\mathrm{i}) \in \mathbb{Z}[\mathrm{i}]^4$

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For $d \in \mathbb{N}$

$$\mathcal{D}_d \;:= \{\mathbf{r} \in \mathbb{R}^d \mid \text{ each orbit of } \tau_{\mathbf{r}} \text{ is ultimately periodic}\}$$
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Orbit of 2 ultimately periodic!
$r \in \mathcal{G}_1$?

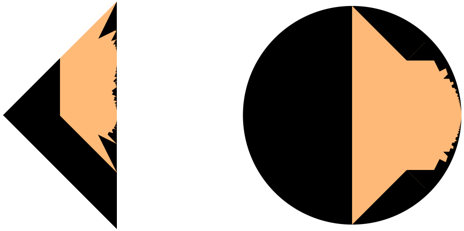

Figure: $\mathcal{D}_2^{(0)}$ in $\mathcal{D}_2$ and $\mathcal{G}_1^{(0)}$ in $\mathcal{G}_1$

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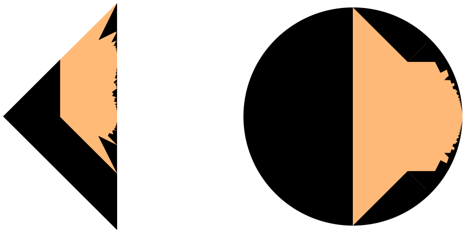

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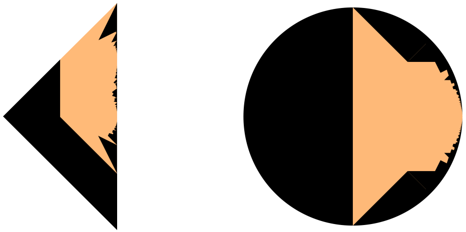

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Relation between SRS, β-Expansions and Canonical Number Systems

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Then $\mathcal{A} := \{0, 1, \ldots, \lfloor \beta \rfloor\}$ is called the set of digits, as every $\gamma \in [0, \infty)$ can be represented uniquely in the form

$\gamma = a_m \beta^m + a_{m-1} \beta^{m-1} + \cdots$
(greedy expansion of γ with respect to β)

with $m \in \mathbb{Z}$ and $a_i \in \mathcal{A}$, such that

$$0 \leq \gamma - \sum_{i=k}^{m} a_i \beta^i < \beta^k$$

holds for all $k \leq m$.

Motivation - Relation to β-Expansions

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Example:

$\beta = \varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots$ $(\Rightarrow \mathcal{A} = \{0,1\})$
$\gamma = \frac{5}{\varphi} - \frac{11}{\varphi^3} = 0.49342219125\ldots$

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Example:

$\beta = \varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots \; (\Rightarrow \mathcal{A} = \{0,1\})$
$\gamma = \frac{5}{\varphi} - \frac{11}{\varphi^3} = 0.49342219125\ldots$

$$\begin{aligned}&0.493\ldots\ (0) \xrightarrow{T_\beta} 0.798\ldots\ (1) \xrightarrow{T_\beta} 0.291\ldots\ (0) \xrightarrow{T_\beta} 0.472\ldots\ (0) \xrightarrow{T_\beta} \\ &0.763\ldots\ (1) \xrightarrow{T_\beta} 0.236\ldots\ (0) \xrightarrow{T_\beta} 0.381\ldots\ (0) \xrightarrow{T_\beta} 0.618\ldots\ (1) \xrightarrow{T_\beta} \\ &0\ldots\ (0) \xrightarrow{T_\beta} 0\ldots\ (0)\ \ldots\end{aligned}$$

$$\gamma = 0 \cdot \frac{1}{\beta} + 1 \cdot \frac{1}{\beta^2} + 0 \cdot \frac{1}{\beta^3} + 0 \cdot \frac{1}{\beta^4} + 1 \cdot \frac{1}{\beta^5} + 0 \cdot \frac{1}{\beta^6} + 0 \cdot \frac{1}{\beta^7} + 1 \cdot \frac{1}{\beta^8}$$

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Motivation - Relation to β-Expansions

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Then β has property (F) $\iff$ $(r_0, \ldots, r_{d-2}) \in \mathcal{D}_{d-1}^{(0)}$

A similar relation can be shown for CNS:

Let
$P(X) = X^d + p_{d-1}X^{d-1} + \cdots + p_1 X + p_0 \in \mathbb{Z}[X]$
$\mathcal{R} := \mathbb{Z}[X]/P(X)\mathbb{Z}[X]$
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Motivation - Relation to Canonical Number Systems

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$(P, \mathcal{N})$ is called a CNS, P a CNS polynomial and $\mathcal{N}$ the set of digits if every non-zero element $A(x) \in \mathcal{R}$ can be represented uniquely in the form

$$A(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$$

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Then P is a CNS polynomial $\iff$ $\left(\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \ldots, \frac{p_2}{p_0}, \frac{p_1}{p_0}\right) \in \mathcal{D}_d^{(0)}$

For $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$ let

$$R_{\mathbf{r}} := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -r_1 & -r_2 & \cdots & \cdots & -r_d \end{pmatrix}$$

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Equivalent statements are true for $\mathcal{G}_d$.

For a tuple π of vectors in $\mathbb{Z}^d$ let $\mathcal{P}(\pi)$ denote the set of all $\mathbf{r} \in \mathbb{R}^d$ for which π is a period of $\tau_{\mathbf{r}}$.

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Basic properties of SRS and GSRS

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For $d = 1$ we have:

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$\mathcal{D}_1^{(0)} = [0, 1)$

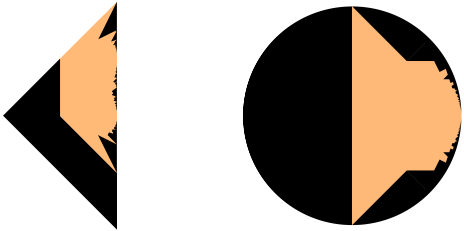
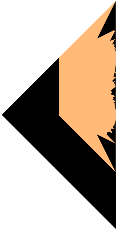
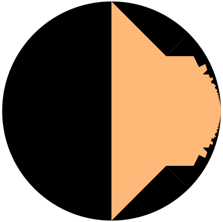


Figure: $\mathcal{D}_2^{(0)}$ in $\mathcal{D}_2$ and $\mathcal{G}_1^{(0)}$ in $\mathcal{G}_1$

$\mathcal{D}_1 = [-1, 1]$, $\mathcal{D}_1^{(0)} = [0, 1)$
$\mathcal{D}_2 \subseteq \{(x, y) \in \mathbb{R}^2 \mid x \geq |y| - 1 \wedge x \leq 1\}$





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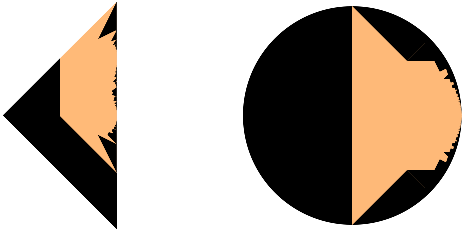

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Hope that characterization of $\mathcal{G}_1^{(0)}$ is of an intermediate level of difficulty.

The real case

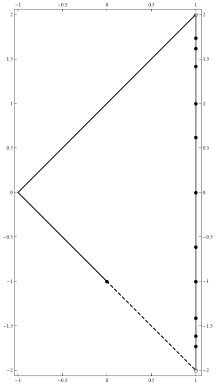

Figure: $\mathcal{D}_2$

Points on right line: $\frac{\pm 1 \pm \sqrt{5}}{2}$, $\pm\sqrt{2}$, $\pm\sqrt{3}$ (quadratic irrational numbers)

Recapitulation:

$$
\begin{aligned}
&r \in \mathbb{C}\\
&\gamma_r : \mathbb{Z}[\mathrm{i}] \mapsto \mathbb{Z}[\mathrm{i}]\\
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Recapitulation:

$r \in \mathbb{C}$
$\gamma_r : \mathbb{Z}[\mathrm{i}] \mapsto \mathbb{Z}[\mathrm{i}]$
$\quad x \to -\lfloor rx \rfloor$

$\{r \in \mathbb{C} \mid |r| < 1\} \subseteq \mathcal{G}_1 \subseteq \{r \in \mathbb{C} \mid |r| \leq 1\}$

An easy argument for the first relation:

$\forall r \in \mathbb{C} : |\gamma_r(x)| < |r||x| + \sqrt{2}$

$|r| < 1 \Rightarrow |\gamma_r(x)| < |x| + \sqrt{2}$
$\quad |x| \geq \frac{\sqrt{2}}{1-|r|} \Rightarrow |r| \leq \frac{|x|-\sqrt{2}}{|x|} \Rightarrow |\gamma_r(x)| < |x|\frac{|x|-\sqrt{2}}{|x|} + \sqrt{2} = |x|$

$\Rightarrow r \in \mathcal{G}_1$
All cycles of r are contained in $\{x \in \mathbb{Z}[\mathrm{i}] \mid |x| < \frac{\sqrt{2}}{1-|r|} + \sqrt{2}\}$

How to decide, whether a given r belongs to $\mathcal{G}_1^{(0)}$ or not?

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Direct or indirect argument possible!

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Methods carry over to SRS and GSRS of arbitrary dimension.

The complex case - Pethő's Loudspeaker

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$n \in \mathbb{Z}$, $x \in \mathbb{R}$

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$$\iff \begin{array}{l} -A = \lfloor xa - yb \rfloor \\ -B = \lfloor xb + ya \rfloor \end{array} \iff \begin{array}{l} 0 \leq xa - yb + A < 1 \\ 0 \leq xb + ya + B < 1 \end{array}$$

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Consider a tuple π of Gaussian integers and calculate the "cut out polygon" $\mathcal{P}(\pi)$.

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Let $\pi = (a_1 + \mathrm{i}b_1, \ldots, a_n + \mathrm{i}b_n)$, then $\mathcal{P}(\pi)$ is given by the following $4n$ integer linear inequalities:

$$0 \leq xa_1 - yb_1 + a_2 < 1$$
$$0 \leq xb_1 + ya_1 + b_2 < 1$$
$$0 \leq xa_2 - yb_2 + a_3 < 1$$
$$0 \leq xb_2 + ya_2 + b_3 < 1$$
$$\vdots$$
$$0 \leq xa_n - yb_n + a_1 < 1$$
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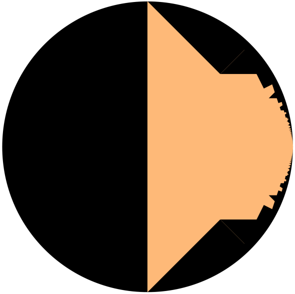

$\mathcal{G}_1^{(0)}$ has empty intersection with the second quadrant.

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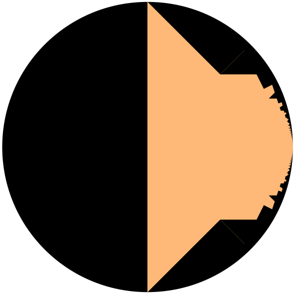

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As $(x_1, \ldots, x_n)$ is a cycle of $r \iff (\mathrm{i}\overline{x_1}, \ldots, \mathrm{i}\overline{x_n})$ is a cycle of $\overline{r}$

$\mathcal{G}_1^{(0)}$ is symmetric and therefore all of its elements have real parts ≥ 0.

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Theorem:
Let $r \in \mathbb{C}$ and $\mathcal{Z}_r$ be the set of all elements of $\mathbb{Z}[\mathrm{i}]$, whose orbits end up in 0.

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Then $r \in \mathcal{G}_1^{(0)}$

The complex case - Pethő's Loudspeaker

We set

$$
\begin{aligned}
S_1 f(x) &= f(x)\\
S_2 f(x) &= -f(-x)\\
S_3 f(x) &= \overline{f(\overline{x})}\\
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Using the definition of γ_r we deduce for $(a,b) \in \mathbb{Z}[\mathrm{i}]^2$

$\gamma_r(a+b) \in \{\gamma_r(a) + S_i \gamma_r(b) \mid i \in \{1, \ldots, 4\}\}$

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Therefore $a + b \in \mathcal{Z}_r$, so $\mathcal{Z}_r = \mathbb{Z}[\mathrm{i}]$ $(\{1, -1, \mathrm{i}, -\mathrm{i}\} \subseteq V)$ $\square$

Find V recursively: $V_1 := \{1, -1, \mathrm{i}, -\mathrm{i}\}$
$V_n := V_{n-1} \cup \{S_i \gamma_r(x) \mid x \in V_{n-1} \wedge i \in \{1, \ldots, 4\}\}$
$V := \cup_{n=1}^{\infty} V_n$

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If $|r| < 1$ then $V \subseteq \{x \in \mathbb{Z}[\mathrm{i}] \mid |x| < \frac{\sqrt{2}}{1-|r|} + \sqrt{2}\}$

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Idea leads to an algorithm by Brunotte, which always calculates $\mathcal{G}_1^{(0)} \cap conv(r_1, \ldots, r_n)$ i.e. the intersection of $\mathcal{G}_1^{(0)}$ with the convex hull of finitely many interior points of $\mathcal{G}_1$ in finitely many steps.

Example: V for $r = \frac{9}{10} + \mathrm{i}\frac{6}{17}$

The complex case - Pethő's Loudspeaker

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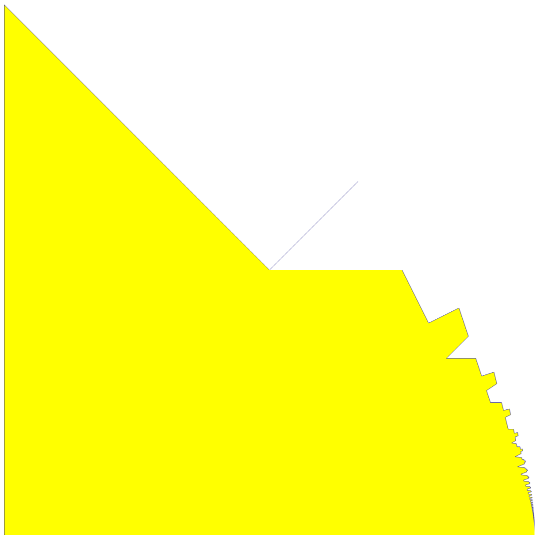
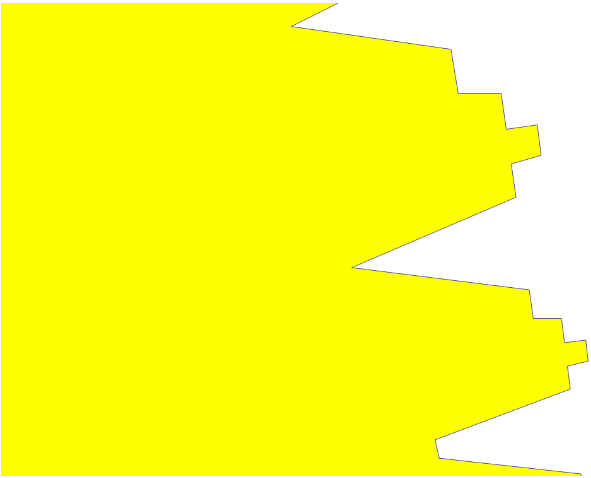
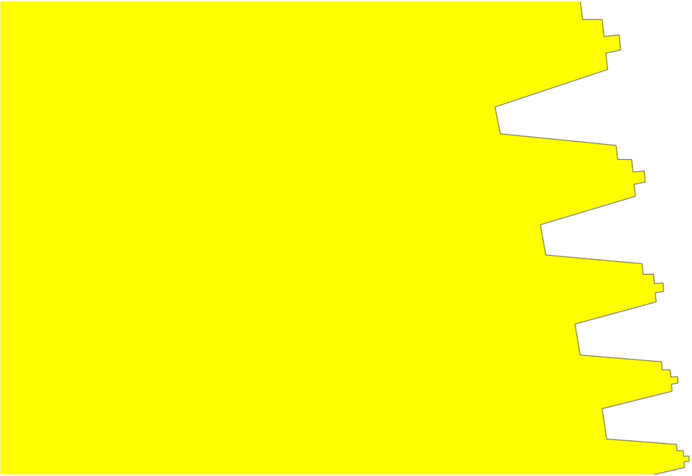
For further considerations let $\mathcal{V}$ denote the set of the $4 \cdot |V|$ arrows on V (images under $S_1\gamma_r, \ldots, S_4\gamma_r$).



Some observations and results





10 Points:

$$z_1(n) = 1 + \frac{-2+\mathrm{i}n}{n^2-2}$$

$$z_2(n) = 1 + \frac{-1+\mathrm{i}(n-1)}{n^2-n-1}$$

$$z_3(n) = 1 + \frac{-1+\mathrm{i}(n-1)}{n^2-n}$$

$$z_4(n) = 1 + \frac{-1+\mathrm{i}n}{n^2}$$

$$z_5(n) = 1 + \frac{-1+\mathrm{i}n}{n^2+1}$$

$$z_6(n) = 1 + \frac{-1+\mathrm{i}(n+1)}{n^2+n+1}$$

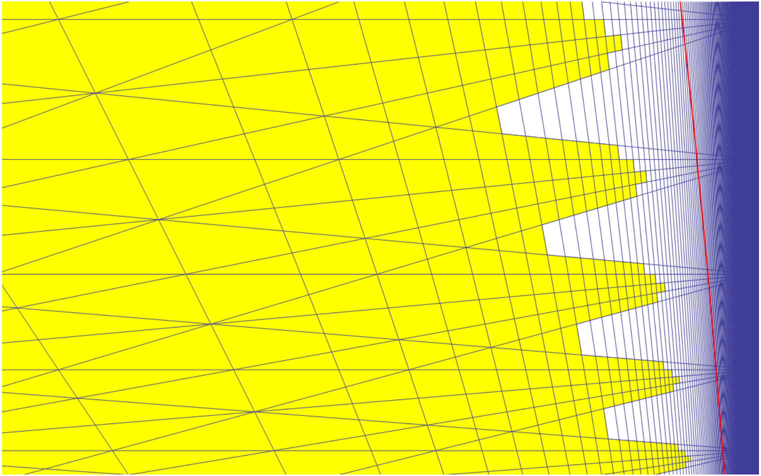
$$z_7(n) = 1 + \frac{-1+\mathrm{i}(n+1)}{n^2+n+2}$$

$$z_8(n) = 1 + \frac{-1+\mathrm{i}n}{n^2+2}$$

$$z_9(n) = 1 + \frac{-1+\mathrm{i}n}{n^2+3}$$

$$z_{10}(n) = 1 + \frac{-2+\mathrm{i}(n+1)}{n^2+n+6}$$

Is $\mathcal{G}_1^{(0)}$ star-shaped?



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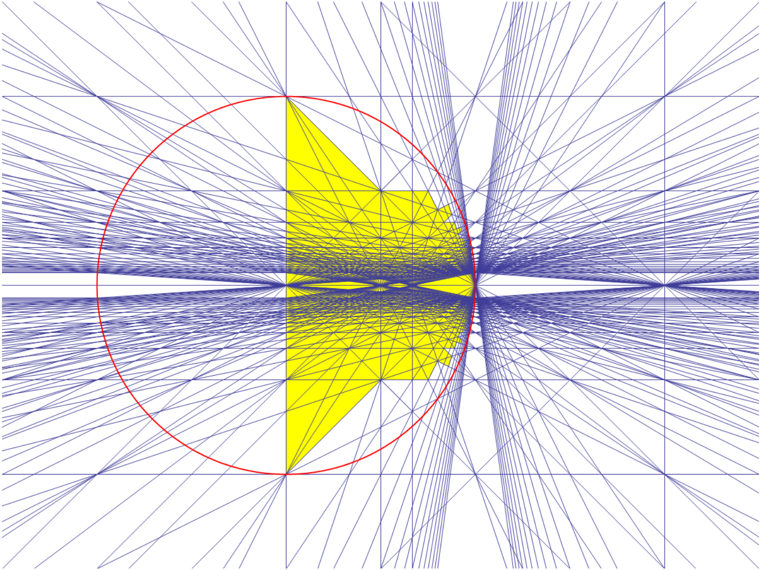


6 families of lines:

$$(0,0) + t(p,1)$$

$$(\tfrac{1}{2},0) + t(p,2)$$

$$(\tfrac{2}{3},0) + t(p,3)$$

$$(1,0) + t(-2,p)$$

$$(2,0) + t(p,-1)$$

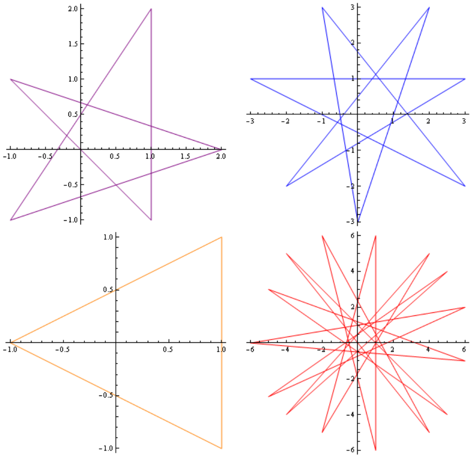
$$(0,\tfrac{1}{p}) + t(1,0)$$

The corresponding polygons of which cycles "touch"

$$\mathcal{G}_1^{(0)} = \mathcal{G}_1 \setminus \bigcup_{\pi \neq 0} \mathcal{P}(\pi)$$

?

Cut outs: 4 families of cycles



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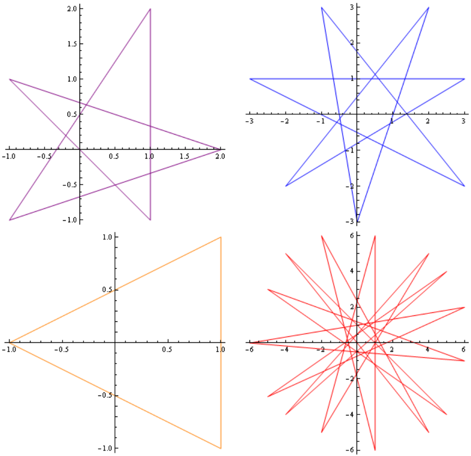
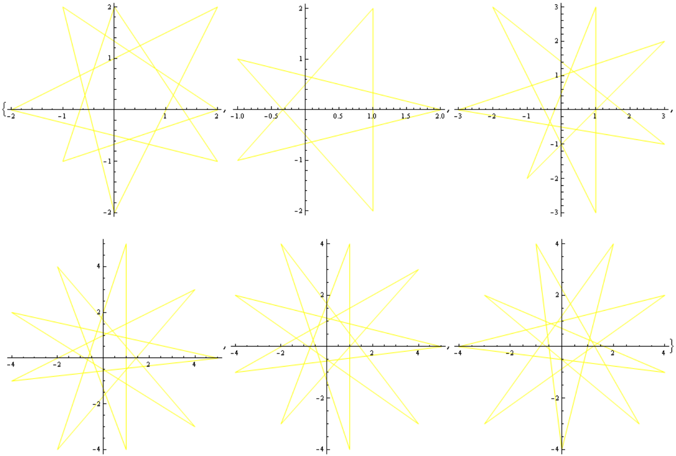
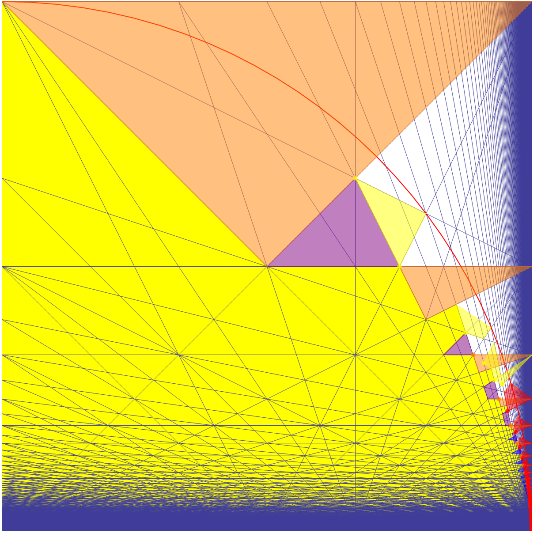
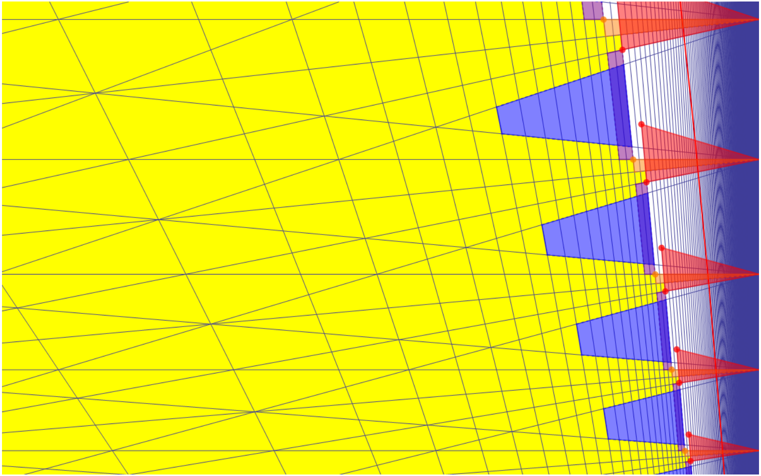


Cut outs: 6 additional cycles





Some observations and results



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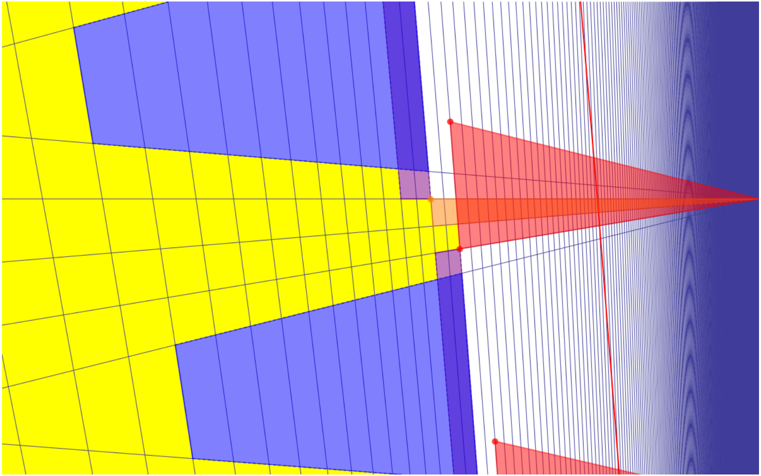

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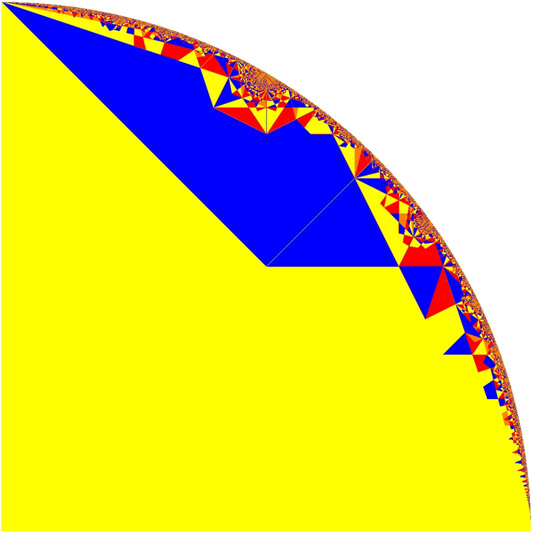

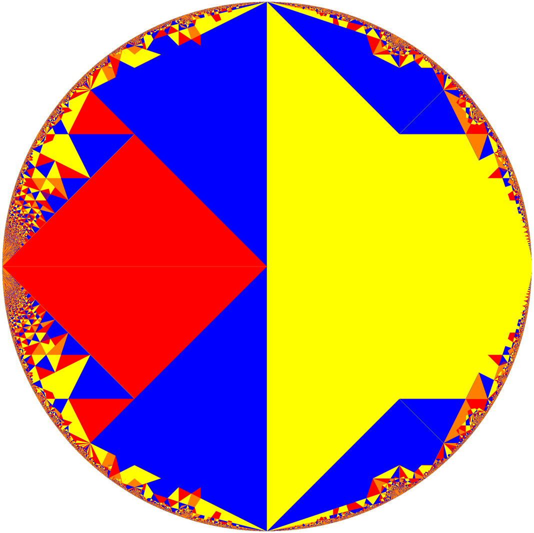

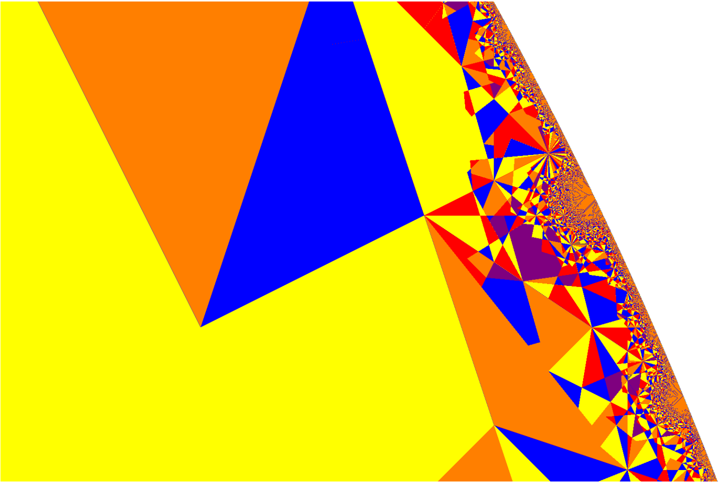

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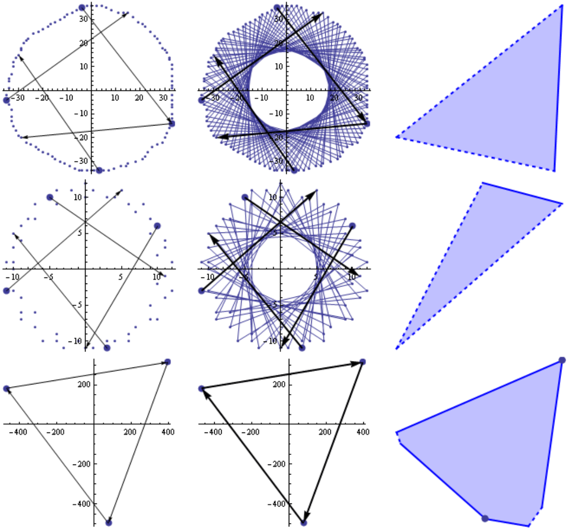

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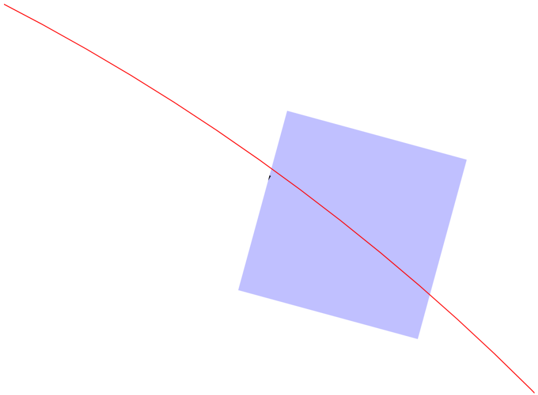

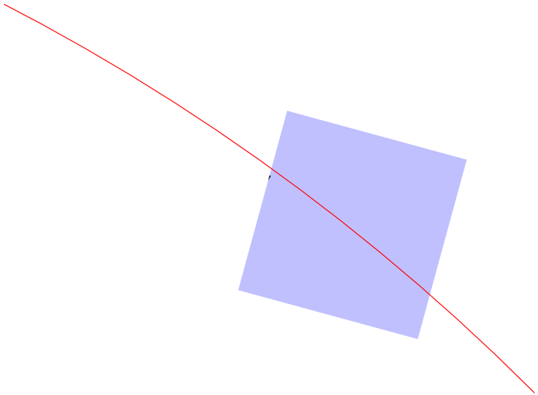

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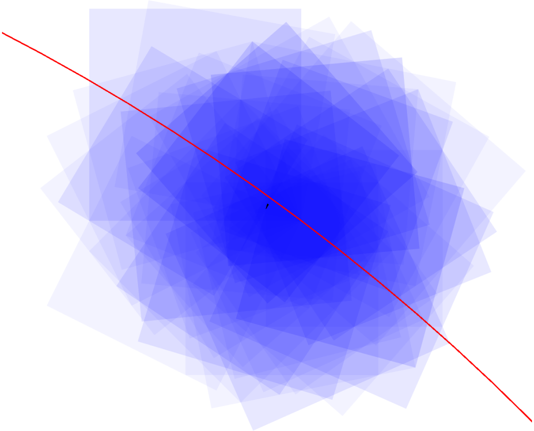
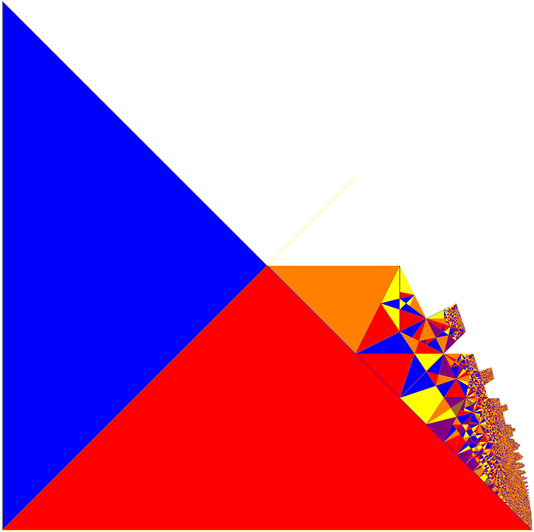


A possible direct proof of the loudspeaker's structure:

Calculating the polygon $\pi(\mathcal{V})$ for a given $\mathcal{V}$
(set of arrows on V).

Some observations and results



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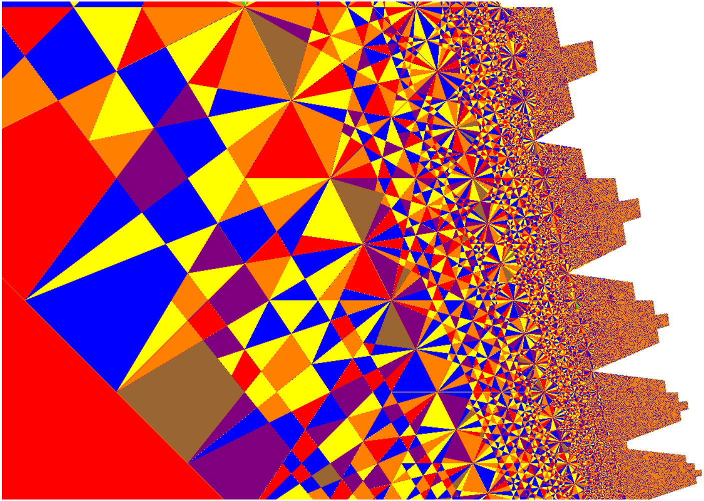

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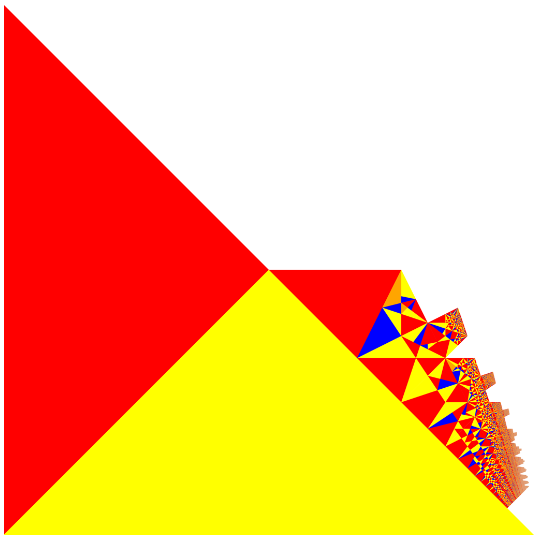
"Automatic prover" searching for polygons.



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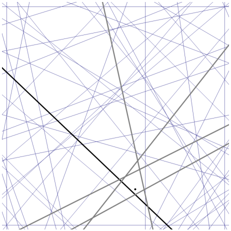
If S is completely covered by polygons, stop.
Otherwise increment n and repeat the steps above.

Facts on the polygons:

The polygons always contain the parameter used finding them. This provides an efficient method for solving the system of linear inequalities.



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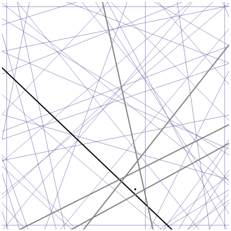

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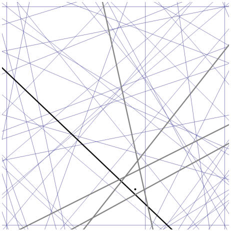
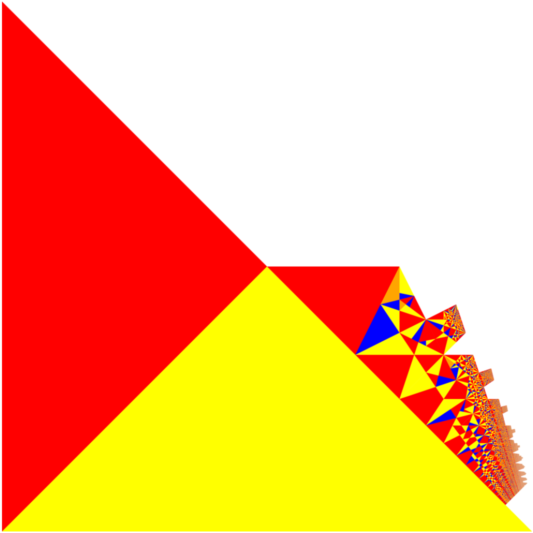
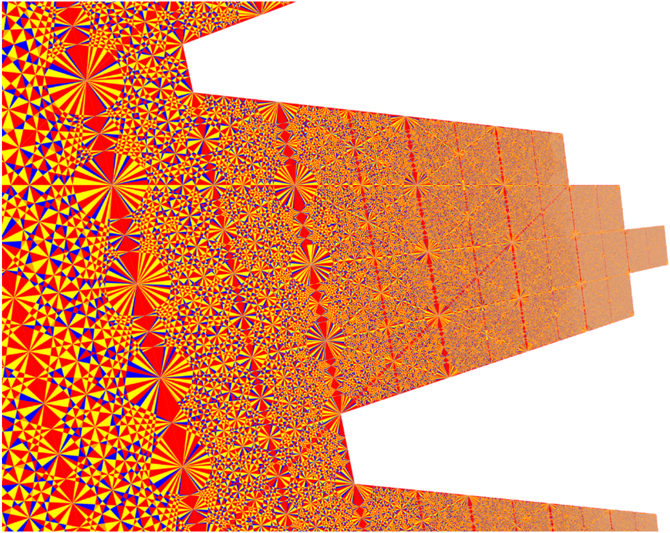
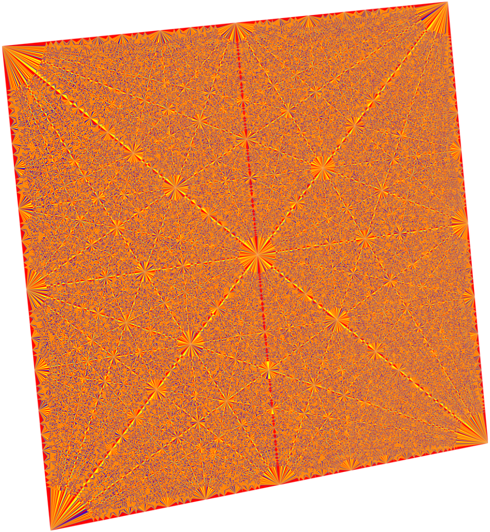


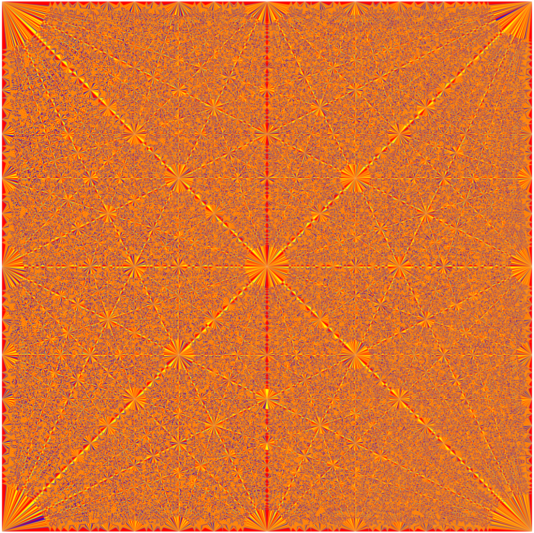
Solved systems involving more than 1.5 billion linear inequalities.

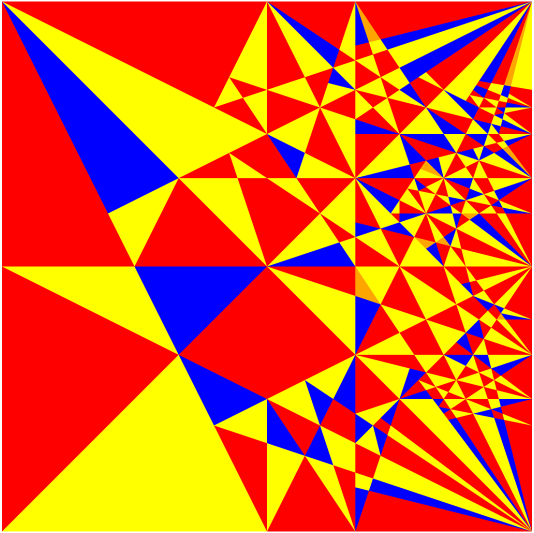
The polygons are either single points, or open line segments, or non-degenerate and open.

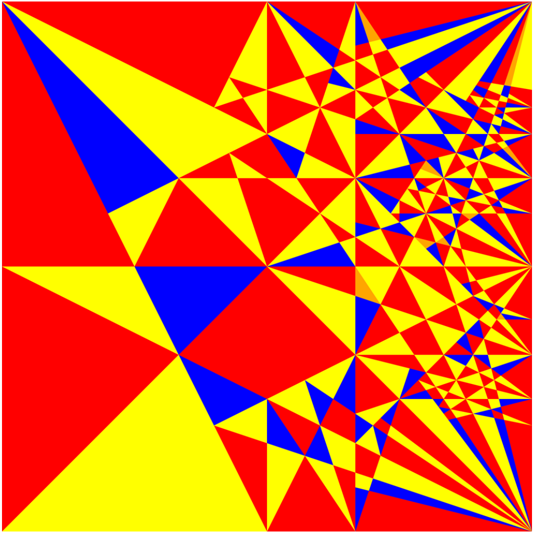


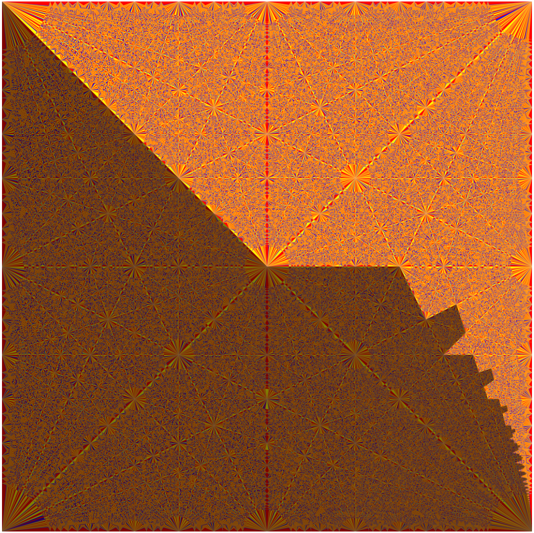


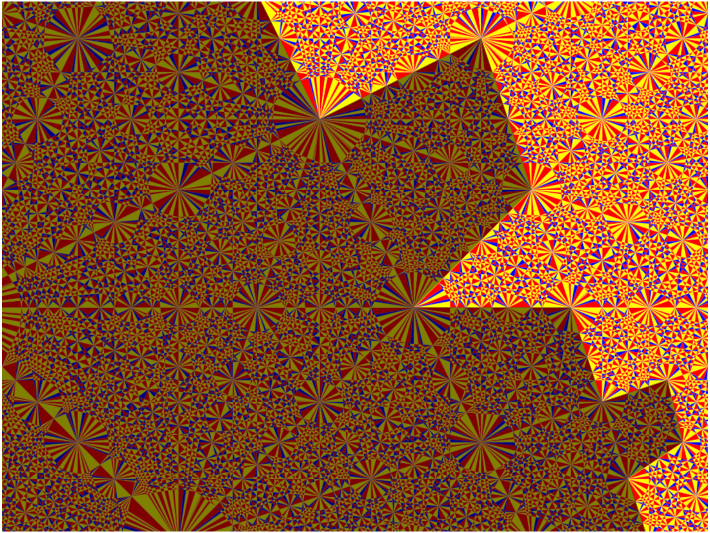


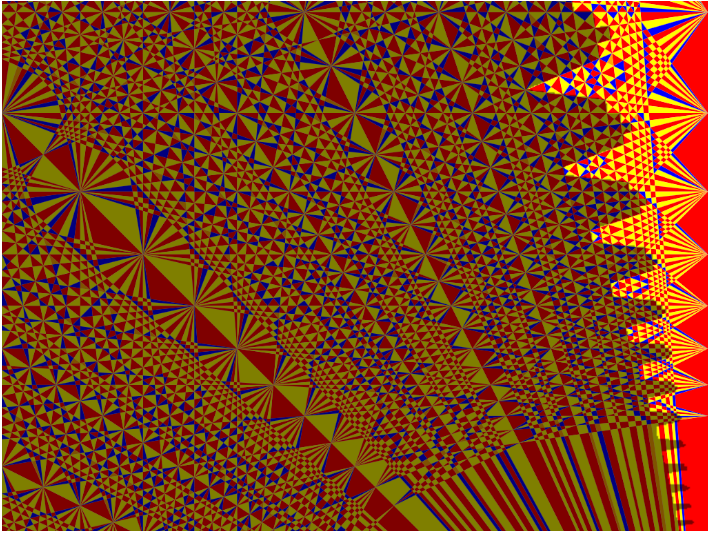












Open questions:

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Which property corresponds to the finiteness property?

Thank you for your attention!