Four years of research

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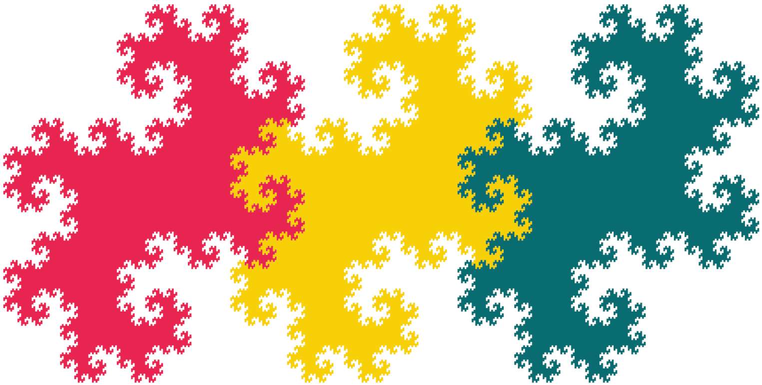
Thank You!



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Research stays in Debrecen (HU), Calgary (CA), Montpellier (FR), and Rennes (FR)
Worked with Attila Pethő, Peter Varga, Vassil Dimitrov, Renate Scheidler, Laurent Imbert, and Anne Siegel

Shift Radix Systems and Their Generalizations

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$\mathcal{D}_d$ and $\mathcal{D}_d^{(0)}$ have a very complicated structure for $d \geq 2$

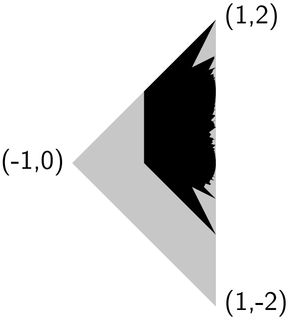
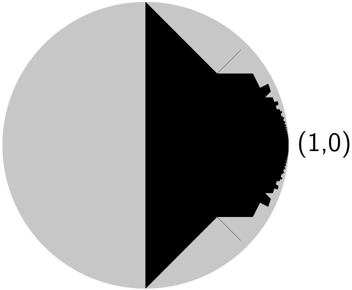


Figure: $\mathcal{D}_2^{(0)}$ in $\mathcal{D}_2$



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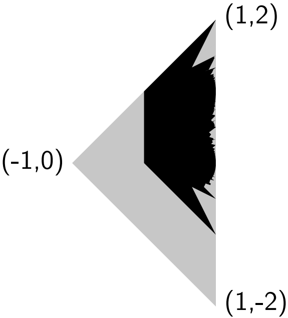

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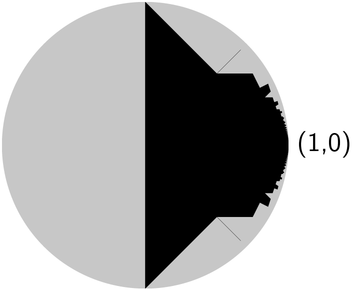

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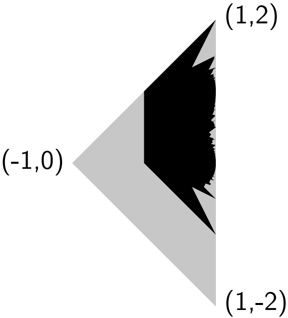
Why?

My thesis in a nutshell



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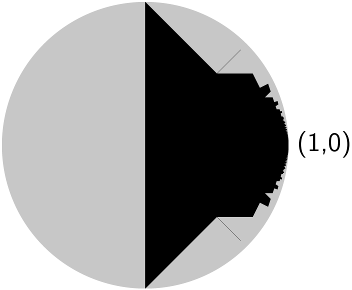

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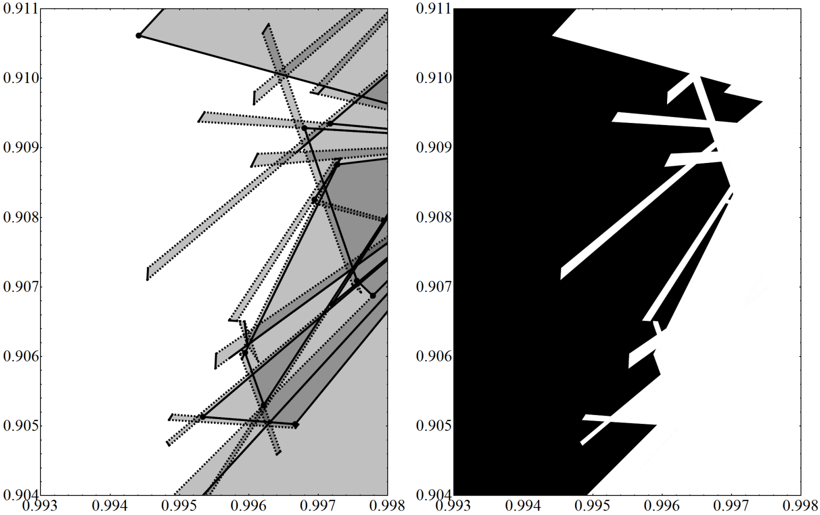
Relation between SRS, β-Expansions, and Canonical Number Systems

Theorem (M.W.):

- $\mathcal{D}_2^{(0)}$ has at least 22 connected components
- The largest connected component of $\mathcal{D}_2^{(0)}$ has at least 3 holes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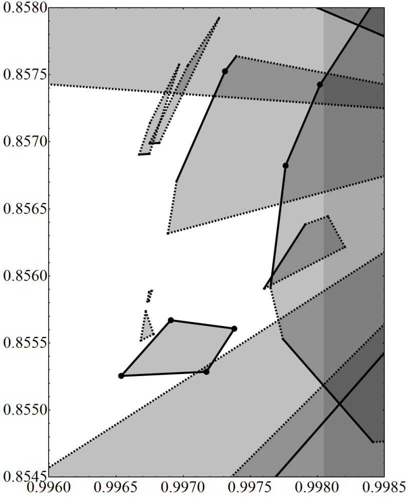

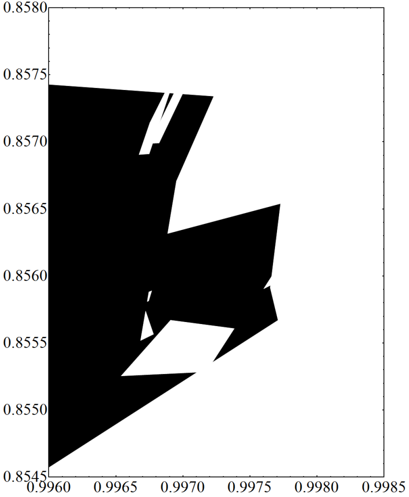

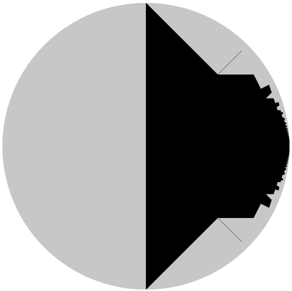

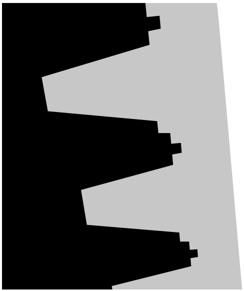

Conjecture by M.W.:

$\mathcal{G}_1^{(0)} = \mathcal{G}_C$ where $\mathcal{G}_C$ is a (neither open nor closed) polygon given by ten infinite sequences of points in $\mathbb{C}$ (and their complex conjugates)

Theorem (M.W.):

$\mathcal{G}_1^{(0)} \subseteq \mathcal{G}_C$

Other inclusion: Settled in large parts

$\mathcal{E}_d^{(\mathbb{R})} := \{(r_0, \dots, r_{d-1}) \in \mathbb{R}^d \mid x^d + x^{d-1} r_{d-1} + \dots + r_0 \text{ contractive}\}$
is called the d-dim. real Schur-Cohn region

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Generalization (Pethő and Akiyama):

$$\mathcal{E}_{d,s}^{(\mathbb{R})} := \{(r_0, \dots, r_{d-1}) \in \mathcal{E}_d^{(\mathbb{R})} \mid x^d + x^{d-1} r_{d-1} + \dots + r_0 \text{ has exactly } 2s \text{ complex roots}\}$$

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$v_d := \lambda_d(\mathcal{E}_d^{(\mathbb{R})})$ (d-dim. Lebesgue measure)
$v_d^{(s)} := \lambda_d(\mathcal{E}_{d,s}^{(\mathbb{R})})$

d	$v_d/v_d^{(0)}$	$v_d^{(1)}/v_d^{(0)}$	$v_d^{(2)}/v_d^{(0)}$	$v_d^{(3)}/v_d^{(0)}$
2	3	2		
3	15	14		
4	175	78	96	
5	3675	418	3256	
6	169785	2244	85620	81920
7	14567553	12156	2173188	12382208
8	2678348673	66428	56138244	1447738880

Table: Values of $v_d/v_d^{(0)}$ and $v_d^{(s)}/v_d^{(0)}$

Conjecture by A. Pethő and S. Akiyama: $v_d^{(s)}/v_d^{(0)} \in \mathbb{Z}$

Theorem (P. Kirschenhofer, M.W.):

$v_d^{(1)}/v_d^{(0)} = \frac{P_d(3)-2d-1}{4}$ where P_d is the d-th Legendre polynomial

In particular: $v_d^{(1)}/v_d^{(0)} \in \mathbb{Z}$

Cleaning up a mess:

$$\frac{v_d^{(1)}}{v_d^{(0)}} = \Bigg(2^{(d-1)(d-2)/2-2}$$

$$\sum_{j=0}^{d-2} \sum_{k=0}^{d-2-j} \Bigg(\frac{(-1)^{d+k} 2^{2d-2-2k-j}}{j!k!(d-2-j-k)!} \prod_{i=1}^{d-2-k-j} \frac{2+(d-2-i-1)/2}{3+(2(d-2)-i-1)/2}$$

$$\frac{\prod_{i=1}^{d-2-k}(1+(d-2-i)/2)\prod_{i=1}^{d-2-k-j}(1+(d-2-i)/2)}{\prod_{i=1}^{d-2-k+d-2-k-j}(2+(2(d-2)-i-1)/2)}$$

$$\frac{1}{\prod_{i=0}^{d-2-1}\binom{2i+1}{i}} \int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^j (y+z+1)^k \, dy \, dz \Bigg)\Bigg) /$$

$$\left(\frac{2^{d(d+1)/2}}{d!} \frac{1}{\prod_{i=0}^{d-1}\binom{2i+1}{i}} \right)$$

$$= \frac{P_d(3) - 2d - 1}{4}$$

Published results:

1) **Characterization algorithms for shift radix systems with finiteness property**, M. Weitzer, 2015, Int. J. Number Theory, 11(1)
2) **On the characterization of Pethő's Loudspeaker**, M. Weitzer, 2015, Publ. Math. Debrecen, 87(1–2)
3) **A number theoretic problem on the distribution of polynomials with bounded roots**, P. Kirschenhofer and M. Weitzer, 2015, Integers, 15(#A10)
4) **On shift radix systems over imaginary quadratic Euclidean domains**, A. Pethő, P. Varga, and M. Weitzer, 2015, Acta Cybernetica, to appear.

Preparing a FWF project application on Collatz-type dynamical systems ($3n + 1$ problem)

Joint work with Robert Tichy:

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Let $u_{K,S}(n;q)$ be the number of representations of algebraic integers α with $\left|N_{K/\mathbb{Q}}(\alpha)\right| \leq q \in \mathbb{R}_{>0}$ that can be written as sums of exactly n S-units of the number field K (S a finite set of places of K that includes the Archimedean ones with $|S| = s+1$)

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Then

$$u(n;q) = \frac{c_{n-1,s}}{n!}\left(\frac{\omega_K(\log(q))^s}{\mathrm{Reg}_{K,S}}\right)^{n-1} + o(\log(q)^{(n-1)s-1+\varepsilon})$$

as $q \to \infty$, where ω_K is the number of roots of unity of K, $\mathrm{Reg}_{K,S}$ the S-regulator of K

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as $q \to \infty$, where ω_K is the number of roots of unity of K, $\mathrm{Reg}_{K,S}$ the S-regulator of K, and $c_{n,s}$ the volume of

$$P_{n,s} := \{(x_{1,1},\ldots,x_{n,s}) \in \mathbb{R}^{ns} \mid g_{n,s}(x_{1,1},\ldots,x_{n,s}) \leq 1\}$$

where

$$g_{n,s}(x_{1,1},\ldots,x_{n,s}) := \sum_{j=1}^s \max\{0, x_{1,j},\ldots,x_{n,j}\} + \max\left\{0, -\sum_{j=1}^s x_{1,j},\ldots,-\sum_{j=1}^s x_{n,j}\right\}$$

(FUCHS, TICHY, ZIEGLER 2009)

Previous results by Barroero, Frei, Fuchs, Tichy, and Ziegler:

Formulas for $c_{n,1}$, $c_{n,2}$, $c_{1,s}$

$s \backslash n$	1	2	3	4	5
1	2	3	4	5	6
2	3	15/4	7/2	45/16	
3	10/3	7/3	55/54		
4	35/12	55/64			
5	21/10				

Table: Values of $c_{n,s}$

My current work

$$g_{n,s}\begin{pmatrix} x_{1,1} & \cdots & x_{1,s} \\ \vdots & & \vdots \\ x_{n,1} & \cdots & x_{n,s} \end{pmatrix} = \max\left\{\begin{matrix} 0 \\ x_{1,1} \\ \vdots \\ x_{n,1} \end{matrix}\right\} + \cdots + \max\left\{\begin{matrix} 0 \\ x_{1,s} \\ \vdots \\ x_{n,s} \end{matrix}\right\} +$$

$$\max\left\{\begin{matrix} 0 \\ -x_{1,1} - \cdots - x_{1,s} \\ \vdots \\ -x_{n,1} - \cdots - x_{n,s} \end{matrix}\right\}$$

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$P_{n,s}$ is a closed non-degenerate convex polytope of dimension ns contained in $[-1,1]^{ns}$ with boundary $\partial(P_{n,s}) = \{\mathbf{x} \in \mathbb{R}^{ns} \mid g_{n,s}(\mathbf{x}) = 1\}$

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$c_{n,s} = \binom{s(n+1)}{s,\dots,s}\frac{1}{(sn)!}$ for all $n, s \in \mathbb{N}$

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Theorem (M.W.):

If $(x_{1,1}, \dots, x_{n,s})$ is a vertex of $P_{n,s}$ then $x_{i,j} \in \{-1, 0, 1\}$ for all i, j

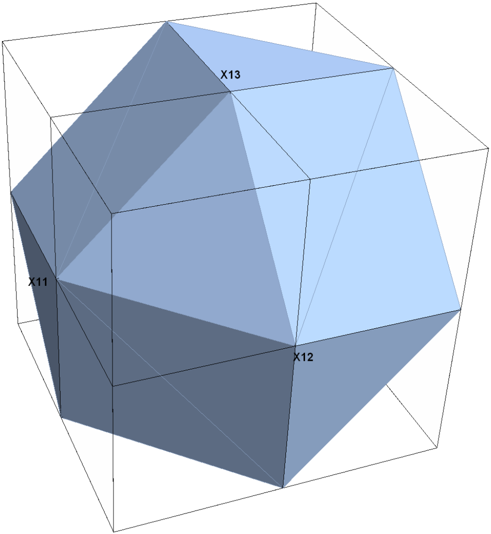

Figure: $P_{1,3}$

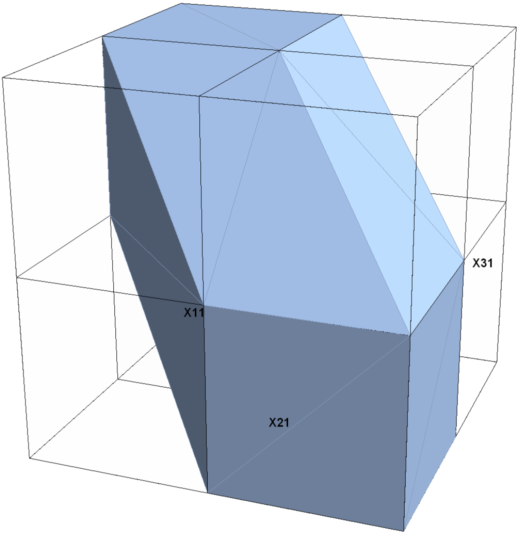

Figure: $P_{3,1}$

More: Complete description of vertices of $P_{n,s}$

Thank you for your attention!