Constrained Triangulations, Volumes of Polytopes, and Unit Equations

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(joint work with Michael Kerber and Robert Tichy)

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Number theoretic motivation: an arithmetic constant

Let	K :	Number field
	S :	Finite set of places of K, containing Archimedean ones
	ω_K :	Number of roots of unity of K
	$\mathrm{Reg}_{K,S}$:	S-regulator of K
	$q \in \mathbb{R}_{>0}$	

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Theorem (Fuchs, Tichy, Ziegler 2009)

$$u_{K,S}(n;q) = \frac{c_{n-1,s}}{n!}\left(\frac{\omega_K \log(q)^s}{\mathrm{Reg}_{K,S}}\right)^{n-1} + o(\log(q)^{(n-1)s-1+\varepsilon}) \quad (q \to \infty)$$

Number theoretic motivation: a family of convex polytopes

$c_{n,s}$ is the volume of

$$P_{n,s} := \{(x_{1,1}, \ldots, x_{n,s}) \in \mathbb{R}^{ns} \mid g_{n,s}(x_{1,1}, \ldots, x_{n,s}) \leq 1\}$$

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where

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Note: We identify $\mathbb{R}^{ns}$ and $\mathbb{R}^{n\times s}$

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- of dimension ns
- contained in $[-1,1]^{ns}$
- with boundary $\partial(P_{n,s}) = \{\mathbf{x} \in \mathbb{R}^{ns} \mid g_{n,s}(\mathbf{x}) = 1\}$

Number theoretic motivation: a family of convex polytopes

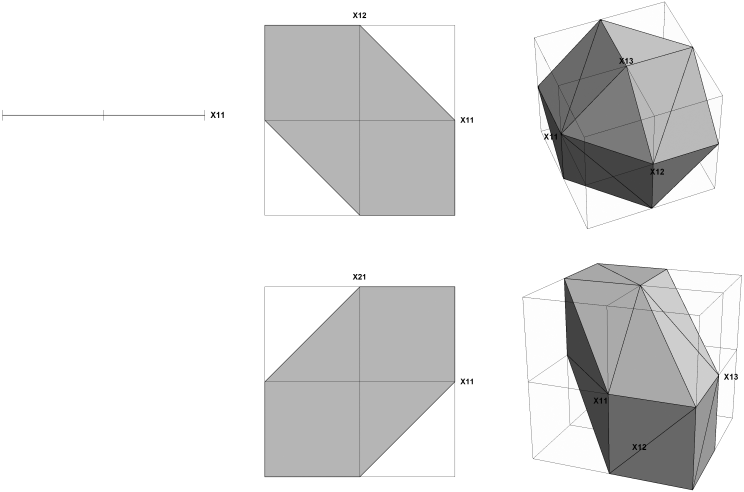

Figure: $P_{1,1}$, $P_{1,2}$, $P_{1,3}$, $P_{2,1}$, $P_{3,1}$

Number theoretic motivation: a family of convex polytopes

$n \backslash s$	1	2	3	4	5
1	2	3	10/3	35/12	21/10
2	3	15/4	7/3	55/64	
3	4	7/2	55/54		
4	5	45/16			
5	6				

Table: Values of $c_{n,s} = \lambda_{ns}(P_{n,s})$

Barroero, Frei, Fuchs, Tichy, and Ziegler: Formulas for $c_{n,1}$, $c_{n,2}$, $c_{1,s}$

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Theorem (Kerber, Tichy, W.)

$$c_{n,s} = \frac{1}{(s!)^{n+1}} \frac{((n+1)s)!}{(ns)!}$$

for all $n, s \in \mathbb{N}$

Geometric motivation: constrained triangulation

In the remaining part of the talk let:

- $d \in \mathbb{N}$
- $\mathcal{P} \subseteq \mathbb{R}^d$ a non-degenerate convex polytope
- $V \subseteq \mathbb{R}^d$ the set of vertices of $\mathcal{P}$
- $U \subseteq V$
- $n := |U|$
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U-spinal triangulation of $\mathcal{P}$: triangulation of $\mathcal{P}$ such that

- every simplex contains U

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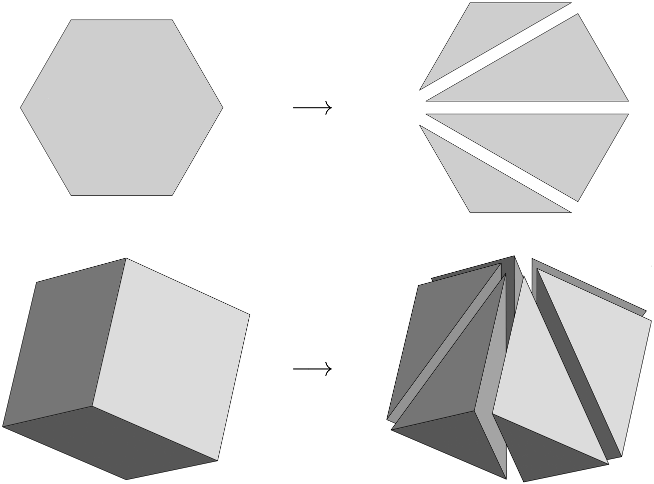

Figure: U-spinal triangulations of a hexagon and a cube

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- $\binom{d}{n-1} \operatorname{vol}(\mathcal{P}) = \operatorname{vol}(U) \operatorname{vol}(\hat{\mathcal{P}})$ (note: $\operatorname{vol}(U) := \operatorname{vol}(\operatorname{conv}(U))$)

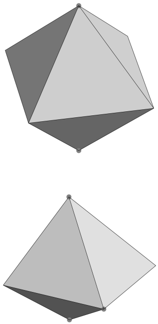
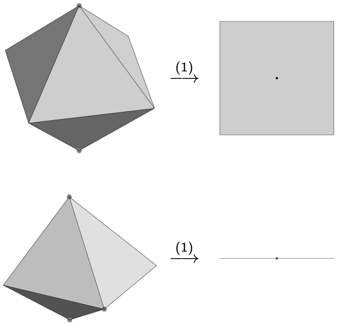


Figure: Two examples of the lifting process

Geometric motivation: constrained triangulation



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(1) Project $\mathcal{P}$ to the orthogonal complement of the subspace spanned by U (prominent dots) to obtain shadow $\hat{\mathcal{P}}$

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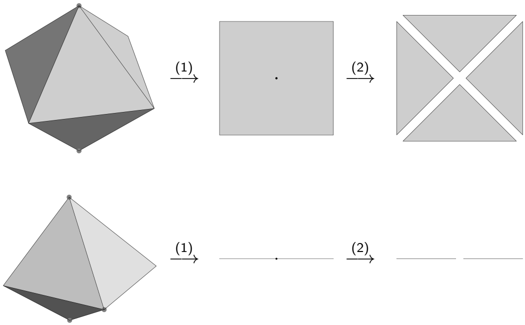

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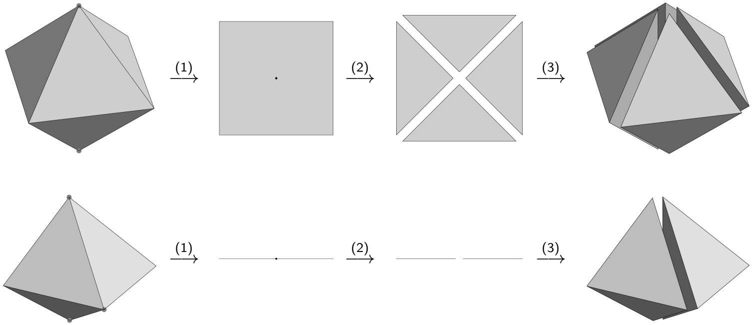

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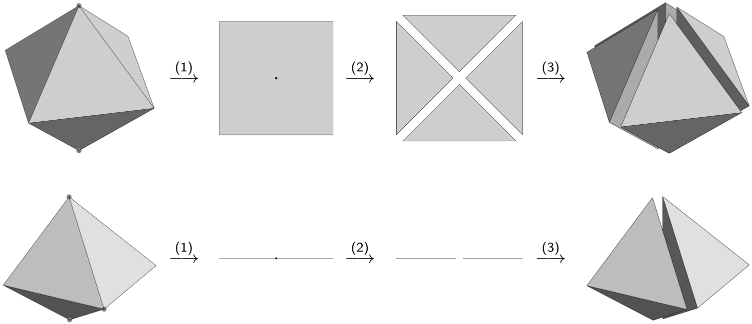

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Note: every facet of $\mathcal{P}$ contains exactly $n-1$ points of U in both examples

Comparison of volumes

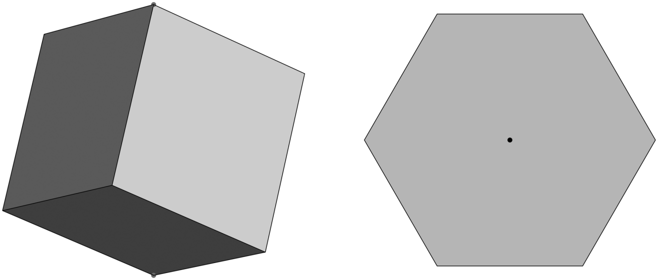

Figure: A cube and its shadow

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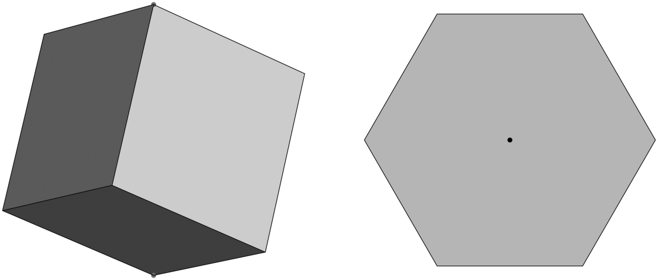

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$$\text{Lifting theorem} \Rightarrow \underbrace{\binom{d}{n-1}}_{\binom{3}{2-1}=3} \underbrace{\operatorname{vol}(\mathcal{P})}_{1} = \underbrace{\operatorname{vol}(U)}_{\sqrt{3}} \underbrace{\operatorname{vol}(\hat{\mathcal{P}})}_{\sqrt{3}}$$

Application of lifting theorem to Everest polytopes

Reminder: Interested in volume $c_{n,s}$ of (n, s)-Everest polytope

$$P_{n,s} := \{(x_{1,1}, \ldots, x_{n,s}) \in \mathbb{R}^{ns} \mid g_{n,s}(x_{1,1}, \ldots, x_{n,s}) \leq 1\}$$

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Then $A_{n,s}\Delta_s^{n+1} = P_{n,s}$ and the (n,s)-Everest polytope is the shadow (almost) of the $(n+1,s)$-simplotope $\Delta_s^{n+1} \subseteq \mathbb{R}^{(n+1)s}$.

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and $\Delta_s \;:=\; \operatorname{conv}\{\mathbf{0},(-1,0,\ldots,0),\ldots,(0,\ldots,0,-1)\} \subseteq \mathbb{R}^s$.

Then $A_{n,s}\Delta_s^{n+1} = P_{n,s}$ and the (n,s)-Everest polytope is the shadow (almost) of the $(n+1,s)$-simplotope $\Delta_s^{n+1} \subseteq \mathbb{R}^{(n+1)s}$.

Lifting theorem $\Rightarrow$ $$\underbrace{\binom{d}{n-1}}_{\binom{(n+1)s}{s+1-1}=\frac{((n+1)s)!}{s!(ns)!}} \; \underbrace{\operatorname{vol}(\Delta_s^{n+1})}_{\frac{1}{(s!)^{n+1}}} \;=\; \underbrace{\operatorname{vol}(U)}_{\frac{1}{s!}} \; \operatorname{vol}(P_{n,s})$$

Application of lifting theorem to Everest polytopes

Reminder: Interested in volume $c_{n,s}$ of (n,s)-Everest polytope

$$P_{n,s} \;:=\; \{(x_{1,1},\ldots,x_{n,s}) \in \mathbb{R}^{ns} \mid g_{n,s}(x_{1,1},\ldots,x_{n,s}) \leq 1\}$$

Lemma: Let I_d be the identity matrix of dimension d,

$$A_{n,s} \;:=\; \begin{pmatrix} & -I_s \\ I_{ns} & \vdots \\ & -I_s \end{pmatrix} \in \mathbb{R}^{(ns)\times((n+1)s)},$$

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Thus $\operatorname{vol}(P_{n,s}) = \frac{1}{(s!)^{n+1}}\frac{((n+1)s)!}{(ns)!}$

Lemma: U is a spine of $\mathcal{P}$ if and only if every facet of $\mathcal{P}$ contains at least $n-1$ points in U

Lifting theorem (Kerber, Tichy, W.)
There exists a U-spinal triangulation of $\mathcal{P}$ if and only if U is a spine of $\mathcal{P}$

- The U-spinal triangulations of $\mathcal{P}$ are exactly the lifts of the star-triangulations of $\hat{\mathcal{P}}$ with respect to $\mathbf{0}$ and
- $\binom{d}{n-1} \operatorname{vol}(\mathcal{P}) = \operatorname{vol}(U) \operatorname{vol}(\hat{\mathcal{P}})$

Theorem (Kerber, Tichy, W.)

- $c_{n,s} = \frac{1}{(s!)^{n+1}} \frac{((n+1)s)!}{(ns)!}$ for all $n, s \in \mathbb{N}$

Thank you for your attention!