(Gaussian) Shift Radix Systems - some new characterization results and topological properties

Mario Weitzer

Doctoral Program
Discrete Mathematics

TU & KFU Graz · MU Leoben
AUSTRIA

October 11, 2013

Let $d \in \mathbb{N}$ and $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$

Let $d \in \mathbb{N}$ and $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$

$$\begin{aligned} \tau_{\mathbf{r}} : \mathbb{Z}^d &\mapsto \mathbb{Z}^d \\ \mathbf{x} = (x_1, \ldots, x_d) &\to (x_2, \ldots, x_d, -\lfloor \mathbf{rx} \rfloor) \end{aligned}$$

is called the d - dimensional SRS associated with $\mathbf{r}$ (AKIYAMA et al. 2005)

where $\mathbf{rx} = \sum_{i=1}^{d} r_i x_i$ denotes the scalar product of $\mathbf{r}$ and $\mathbf{x}$
and $\lfloor y \rfloor$ the largest integer less than or equal to some real y. (floor)

Let $d \in \mathbb{N}$ and $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$

$$\begin{aligned} \tau_{\mathbf{r}} : \mathbb{Z}^d &\mapsto \mathbb{Z}^d \\ \mathbf{x} = (x_1, \ldots, x_d) &\to (x_2, \ldots, x_d, -\lfloor \mathbf{rx} \rfloor) \end{aligned}$$

is called the d - dimensional SRS associated with $\mathbf{r}$ (AKIYAMA et al. 2005)

where $\mathbf{rx} = \sum_{i=1}^{d} r_i x_i$ denotes the scalar product of $\mathbf{r}$ and $\mathbf{x}$
and $\lfloor y \rfloor$ the largest integer less than or equal to some real y. (floor)

$$\begin{aligned} \mathcal{D}_d &:= \{\mathbf{r} \in \mathbb{R}^d \mid \text{ each orbit of } \tau_{\mathbf{r}} \text{ is ultimately periodic}\} \\ \mathcal{D}_d^{(0)} &:= \{\mathbf{r} \in \mathbb{R}^d \mid \text{ each orbit of } \tau_{\mathbf{r}} \text{ ends up in } \mathbf{0}\} \end{aligned}$$

Let $d \in \mathbb{N}$ and $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$

$$\begin{aligned} \tau_{\mathbf{r}} : \mathbb{Z}^d &\mapsto \mathbb{Z}^d \\ \mathbf{x} = (x_1, \ldots, x_d) &\to (x_2, \ldots, x_d, -\lfloor \mathbf{rx} \rfloor) \end{aligned}$$

is called the d - dimensional SRS associated with $\mathbf{r}$ (AKIYAMA et al. 2005)

where $\mathbf{rx} = \sum_{i=1}^{d} r_i x_i$ denotes the scalar product of $\mathbf{r}$ and $\mathbf{x}$ and $\lfloor y \rfloor$ the largest integer less than or equal to some real y. (floor)

$$\begin{aligned} \mathcal{D}_d &:= \{\mathbf{r} \in \mathbb{R}^d \mid \text{ each orbit of } \tau_{\mathbf{r}} \text{ is ultimately periodic}\} \\ \mathcal{D}_d^{(0)} &:= \{\mathbf{r} \in \mathbb{R}^d \mid \text{ each orbit of } \tau_{\mathbf{r}} \text{ ends up in } \mathbf{0}\} \end{aligned}$$

Elements of $\mathcal{D}_d^{(0)}$ are said to have the finiteness property.

Let $d \in \mathbb{N}$ and $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{R}^d$

$$\begin{aligned} \tau_{\mathbf{r}} : \mathbb{Z}^d &\mapsto \mathbb{Z}^d \\ \mathbf{x} = (x_1, \dots, x_d) &\to (x_2, \dots, x_d, -\lfloor \mathbf{r}\mathbf{x} \rfloor) \end{aligned}$$

is called the d - dimensional SRS associated with $\mathbf{r}$ (AKIYAMA et al. 2005)

where $\mathbf{r}\mathbf{x} = \sum_{i=1}^{d} r_i x_i$ denotes the scalar product of $\mathbf{r}$ and $\mathbf{x}$
and $\lfloor y \rfloor$ the largest integer less than or equal to some real y. (floor)

$$\begin{aligned} \mathcal{D}_d &:= \{\mathbf{r} \in \mathbb{R}^d \mid \text{ each orbit of } \tau_{\mathbf{r}} \text{ is ultimately periodic}\} \\ \mathcal{D}_d^{(0)} &:= \{\mathbf{r} \in \mathbb{R}^d \mid \text{ each orbit of } \tau_{\mathbf{r}} \text{ ends up in } \mathbf{0}\} \end{aligned}$$

Elements of $\mathcal{D}_d^{(0)}$ are said to have the finiteness property.

$\mathcal{D}_d^{(0)} \subseteq \mathcal{D}_d$

Analogously let $d \in \mathbb{N}$ and $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{C}^d$

Analogously let $d \in \mathbb{N}$ and $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{C}^d$

$$\begin{aligned} \gamma_{\mathbf{r}} : \mathbb{Z}[\mathrm{i}]^d &\mapsto \mathbb{Z}[\mathrm{i}]^d \\ \mathbf{x} = (x_1, \ldots, x_d) &\to (x_2, \ldots, x_d, -\lfloor \mathbf{rx} \rfloor) \end{aligned}$$

is called the d - dimensional GSRS associated with $\mathbf{r}$

where $\mathbf{rx} = \sum_{i=1}^{d} r_i x_i$ denotes the scalar product of $\mathbf{r}$ and $\mathbf{x}$ and $\lfloor y \rfloor := \lfloor \Re(y) \rfloor + \mathrm{i} \lfloor \Im(y) \rfloor$.

Analogously let $d \in \mathbb{N}$ and $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{C}^d$

$$\gamma_{\mathbf{r}} : \mathbb{Z}[\mathrm{i}]^d \mapsto \mathbb{Z}[\mathrm{i}]^d$$
$$\mathbf{x} = (x_1, \ldots, x_d) \to (x_2, \ldots, x_d, -\lfloor \mathbf{rx} \rfloor)$$

is called the d - dimensional GSRS associated with $\mathbf{r}$

where $\mathbf{rx} = \sum_{i=1}^{d} r_i x_i$ denotes the scalar product of $\mathbf{r}$ and $\mathbf{x}$ and $\lfloor y \rfloor := \lfloor \Re(y) \rfloor + \mathrm{i} \lfloor \Im(y) \rfloor$.

$$\mathcal{G}_d := \{\mathbf{r} \in \mathbb{C}^d \mid \text{ each orbit of } \gamma_{\mathbf{r}} \text{ is ultimately periodic}\}$$
$$\mathcal{G}_d^{(0)} := \{\mathbf{r} \in \mathbb{C}^d \mid \text{ each orbit of } \gamma_{\mathbf{r}} \text{ ends up in } \mathbf{0}\}$$

Analogously let $d \in \mathbb{N}$ and $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{C}^d$

$$\gamma_{\mathbf{r}} : \mathbb{Z}[\mathrm{i}]^d \mapsto \mathbb{Z}[\mathrm{i}]^d$$
$$\mathbf{x} = (x_1, \ldots, x_d) \to (x_2, \ldots, x_d, -\lfloor \mathbf{rx} \rfloor)$$

is called the d - dimensional GSRS associated with $\mathbf{r}$

where $\mathbf{rx} = \sum_{i=1}^{d} r_i x_i$ denotes the scalar product of $\mathbf{r}$ and $\mathbf{x}$ and $\lfloor y \rfloor := \lfloor \Re(y) \rfloor + \mathrm{i} \lfloor \Im(y) \rfloor$.

$$\mathcal{G}_d := \{\mathbf{r} \in \mathbb{C}^d \mid \text{ each orbit of } \gamma_{\mathbf{r}} \text{ is ultimately periodic}\}$$
$$\mathcal{G}_d^{(0)} := \{\mathbf{r} \in \mathbb{C}^d \mid \text{ each orbit of } \gamma_{\mathbf{r}} \text{ ends up in } \mathbf{0}\}$$

Elements of $\mathcal{G}_d^{(0)}$ are said to have the finiteness property.

Analogously let $d \in \mathbb{N}$ and $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{C}^d$

$$\begin{aligned} \gamma_{\mathbf{r}} : \mathbb{Z}[\mathrm{i}]^d &\mapsto \mathbb{Z}[\mathrm{i}]^d \\ \mathbf{x} = (x_1, \ldots, x_d) &\to (x_2, \ldots, x_d, -\lfloor \mathbf{rx} \rfloor) \end{aligned}$$

is called the d - dimensional GSRS associated with $\mathbf{r}$

where $\mathbf{rx} = \sum_{i=1}^{d} r_i x_i$ denotes the scalar product of $\mathbf{r}$ and $\mathbf{x}$ and $\lfloor y \rfloor := \lfloor \Re(y) \rfloor + \mathrm{i} \lfloor \Im(y) \rfloor$.

$$\begin{aligned} \mathcal{G}_d &:= \{\mathbf{r} \in \mathbb{C}^d \mid \text{ each orbit of } \gamma_{\mathbf{r}} \text{ is ultimately periodic}\} \\ \mathcal{G}_d^{(0)} &:= \{\mathbf{r} \in \mathbb{C}^d \mid \text{ each orbit of } \gamma_{\mathbf{r}} \text{ ends up in } \mathbf{0}\} \end{aligned}$$

Elements of $\mathcal{G}_d^{(0)}$ are said to have the finiteness property.

$\mathcal{G}_d^{(0)} \subseteq \mathcal{G}_d$

Relation between SRS and GSRS

SRS	GSRS
$\tau_{\mathbf{r}} : \mathbb{Z}^d \mapsto \mathbb{Z}^d$ $(\mathbf{r} \in \mathbb{R}^d)$	$\gamma_{\mathbf{r}} : \mathbb{Z}[\mathrm{i}]^d \mapsto \mathbb{Z}[\mathrm{i}]^d$ $(\mathbf{r} \in \mathbb{C}^d)$
$\mathcal{D}_d$	$\mathcal{G}_d$
$\mathcal{D}_d^{(0)}$	$\mathcal{G}_d^{(0)}$

SRS	GSRS
$\tau_{\mathbf{r}} : \mathbb{Z}^d \mapsto \mathbb{Z}^d$ ($\mathbf{r} \in \mathbb{R}^d$)	$\gamma_{\mathbf{r}} : \mathbb{Z}[\mathrm{i}]^d \mapsto \mathbb{Z}[\mathrm{i}]^d$ ($\mathbf{r} \in \mathbb{C}^d$)
$\mathcal{D}_d$	$\mathcal{G}_d$
$\mathcal{D}_d^{(0)}$	$\mathcal{G}_d^{(0)}$

In particular for $d = 2 \leftrightarrow d = 1$:

Identify $\mathbb{C} \leftrightarrow \mathbb{R}^2$ and $\mathbb{Z}[\mathrm{i}] \leftrightarrow \mathbb{Z}^2$

$$\begin{aligned}\tau_{(r,s)} : \mathbb{Z}^2 &\mapsto \mathbb{Z}^2 \\ (a,b) &\to (b, -\lfloor ra + sb \rfloor)\end{aligned}$$

$$\begin{aligned}\gamma_{(r,s)} : \mathbb{Z}^2 &\mapsto \mathbb{Z}^2 \\ (a,b) &\to (-\lfloor ra - sb \rfloor, -\lfloor rb + sa \rfloor)\end{aligned}$$

Example:

$d = 2$
$\mathbf{r} = (\frac{9}{10}, \frac{13}{10}) \in \mathbb{R}^2$

$\tau_{\mathbf{r}}((x_1, x_2)) = (x_2, -\lfloor \mathbf{rx} \rfloor)$

Example:

$d = 2$
$\mathbf{r} = (\frac{9}{10}, \frac{13}{10}) \in \mathbb{R}^2$

$\tau_{\mathbf{r}}((x_1, x_2)) = (x_2, -\lfloor \mathbf{rx} \rfloor)$

$(0, 3)$

Example:

$d = 2$
$\mathbf{r} = (\frac{9}{10}, \frac{13}{10}) \in \mathbb{R}^2$

$\tau_{\mathbf{r}}((x_1, x_2)) = (x_2, -\lfloor \mathbf{rx} \rfloor)$

$(0,3) \xrightarrow{\tau_{\mathbf{r}}} (3,-3)$

Example:

$d = 2$
$\mathbf{r} = (\frac{9}{10}, \frac{13}{10}) \in \mathbb{R}^2$

$\tau_{\mathbf{r}}((x_1, x_2)) = (x_2, -\lfloor \mathbf{rx} \rfloor)$

$(0,3) \xrightarrow{\tau_{\mathbf{r}}} (3,-3) \xrightarrow{\tau_{\mathbf{r}}} (-3,2)$

Example:

$d = 2$
$\mathbf{r} = (\frac{9}{10}, \frac{13}{10}) \in \mathbb{R}^2$

$\tau_{\mathbf{r}}((x_1, x_2)) = (x_2, -\lfloor \mathbf{rx} \rfloor)$

$(0,3) \xrightarrow{\tau_{\mathbf{r}}} (3,-3) \xrightarrow{\tau_{\mathbf{r}}} (-3,2) \xrightarrow{\tau_{\mathbf{r}}} (2,1)$

Example:

$d = 2$
$\mathbf{r} = (\frac{9}{10}, \frac{13}{10}) \in \mathbb{R}^2$

$\tau_{\mathbf{r}}((x_1, x_2)) = (x_2, -\lfloor \mathbf{rx} \rfloor)$

$(0, 3) \xrightarrow{\tau_{\mathbf{r}}} (3, -3) \xrightarrow{\tau_{\mathbf{r}}} (-3, 2) \xrightarrow{\tau_{\mathbf{r}}} (2, 1) \xrightarrow{\tau_{\mathbf{r}}} (1, -3)$

Example:

$d = 2$
$\mathbf{r} = (\frac{9}{10}, \frac{13}{10}) \in \mathbb{R}^2$

$\tau_{\mathbf{r}}((x_1, x_2)) = (x_2, -\lfloor \mathbf{rx} \rfloor)$

$(0,3) \xrightarrow{\tau_{\mathbf{r}}} (3,-3) \xrightarrow{\tau_{\mathbf{r}}} (-3,2) \xrightarrow{\tau_{\mathbf{r}}} (2,1) \xrightarrow{\tau_{\mathbf{r}}} (1,-3) \xrightarrow{\tau_{\mathbf{r}}} (-3,3)$

Example:

$d = 2$
$\mathbf{r} = (\frac{9}{10}, \frac{13}{10}) \in \mathbb{R}^2$

$\tau_{\mathbf{r}}((x_1, x_2)) = (x_2, -\lfloor \mathbf{rx} \rfloor)$

$(0,3) \xrightarrow{\tau_{\mathbf{r}}} (3,-3) \xrightarrow{\tau_{\mathbf{r}}} (-3,2) \xrightarrow{\tau_{\mathbf{r}}} (2,1) \xrightarrow{\tau_{\mathbf{r}}} (1,-3) \xrightarrow{\tau_{\mathbf{r}}} (-3,3)$
$\xrightarrow{\tau_{\mathbf{r}}} (3,-1)$

Example:

$d = 2$
$\mathbf{r} = (\frac{9}{10}, \frac{13}{10}) \in \mathbb{R}^2$

$\tau_{\mathbf{r}}((x_1, x_2)) = (x_2, -\lfloor \mathbf{rx} \rfloor)$

$(0,3) \xrightarrow{\tau_{\mathbf{r}}} (3,-3) \xrightarrow{\tau_{\mathbf{r}}} (-3,2) \xrightarrow{\tau_{\mathbf{r}}} (2,1) \xrightarrow{\tau_{\mathbf{r}}} (1,-3) \xrightarrow{\tau_{\mathbf{r}}} (-3,3)$
$\xrightarrow{\tau_{\mathbf{r}}} (3,-1) \xrightarrow{\tau_{\mathbf{r}}} (-1,-1)$

Example:

$d = 2$
$\mathbf{r} = (\frac{9}{10}, \frac{13}{10}) \in \mathbb{R}^2$

$\tau_{\mathbf{r}}((x_1, x_2)) = (x_2, -\lfloor \mathbf{r}\mathbf{x} \rfloor)$

$(0,3) \xrightarrow{\tau_{\mathbf{r}}} (3,-3) \xrightarrow{\tau_{\mathbf{r}}} (-3,2) \xrightarrow{\tau_{\mathbf{r}}} (2,1) \xrightarrow{\tau_{\mathbf{r}}} (1,-3) \xrightarrow{\tau_{\mathbf{r}}} (-3,3)$
$\xrightarrow{\tau_{\mathbf{r}}} (3,-1) \xrightarrow{\tau_{\mathbf{r}}} (-1,-1) \xrightarrow{\tau_{\mathbf{r}}} (-1,3)$

Example:

$d = 2$
$\mathbf{r} = (\frac{9}{10}, \frac{13}{10}) \in \mathbb{R}^2$

$\tau_{\mathbf{r}}((x_1, x_2)) = (x_2, -\lfloor \mathbf{r}\mathbf{x} \rfloor)$

$(0,3) \xrightarrow{\tau_{\mathbf{r}}} (3,-3) \xrightarrow{\tau_{\mathbf{r}}} (-3,2) \xrightarrow{\tau_{\mathbf{r}}} (2,1) \xrightarrow{\tau_{\mathbf{r}}} (1,-3) \xrightarrow{\tau_{\mathbf{r}}} (-3,3)$
$\xrightarrow{\tau_{\mathbf{r}}} (3,-1) \xrightarrow{\tau_{\mathbf{r}}} (-1,-1) \xrightarrow{\tau_{\mathbf{r}}} (-1,3) \xrightarrow{\tau_{\mathbf{r}}} (3,-3)$

Example:

$d = 2$
$\mathbf{r} = (\frac{9}{10}, \frac{13}{10}) \in \mathbb{R}^2$

$\tau_{\mathbf{r}}((x_1, x_2)) = (x_2, -\lfloor \mathbf{r}\mathbf{x} \rfloor)$

$(0,3) \xrightarrow{\tau_{\mathbf{r}}} (3,-3) \xrightarrow{\tau_{\mathbf{r}}} (-3,2) \xrightarrow{\tau_{\mathbf{r}}} (2,1) \xrightarrow{\tau_{\mathbf{r}}} (1,-3) \xrightarrow{\tau_{\mathbf{r}}} (-3,3)$
$\xrightarrow{\tau_{\mathbf{r}}} (3,-1) \xrightarrow{\tau_{\mathbf{r}}} (-1,-1) \xrightarrow{\tau_{\mathbf{r}}} (-1,3) \xrightarrow{\tau_{\mathbf{r}}} (3,-3) \xrightarrow{\tau_{\mathbf{r}}} (-3,2)$

Example:

$d = 2$
$\mathbf{r} = (\frac{9}{10}, \frac{13}{10}) \in \mathbb{R}^2$

$\tau_{\mathbf{r}}((x_1, x_2)) = (x_2, -\lfloor \mathbf{r}\mathbf{x} \rfloor)$

$(0,3) \xrightarrow{\tau_{\mathbf{r}}} (3,-3) \xrightarrow{\tau_{\mathbf{r}}} (-3,2) \xrightarrow{\tau_{\mathbf{r}}} (2,1) \xrightarrow{\tau_{\mathbf{r}}} (1,-3) \xrightarrow{\tau_{\mathbf{r}}} (-3,3)$
$\xrightarrow{\tau_{\mathbf{r}}} (3,-1) \xrightarrow{\tau_{\mathbf{r}}} (-1,-1) \xrightarrow{\tau_{\mathbf{r}}} (-1,3) \xrightarrow{\tau_{\mathbf{r}}} (3,-3) \xrightarrow{\tau_{\mathbf{r}}} (-3,2)$

Orbit of $(0,3)$ ultimately periodic!
$\mathbf{r} \in \mathcal{D}_2$?

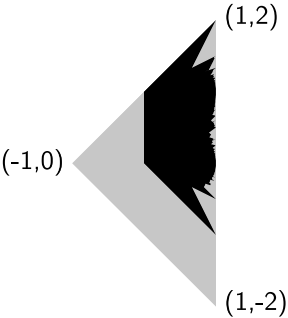


Figure: $\mathcal{D}_2^{(0)}$ in $\mathcal{D}_2$

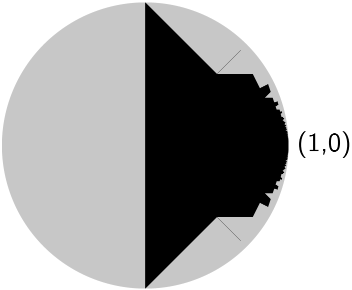


Figure: $\mathcal{G}_1^{(0)}$ in $\mathcal{G}_1$

Interested in $\mathcal{D}_d^{(0)}$ and $\mathcal{G}_d^{(0)}$

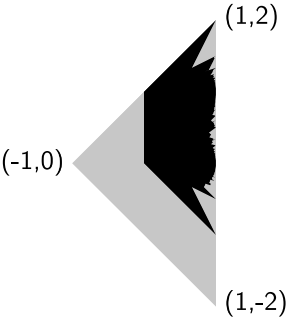


Figure: $\mathcal{D}_2^{(0)}$ in $\mathcal{D}_2$

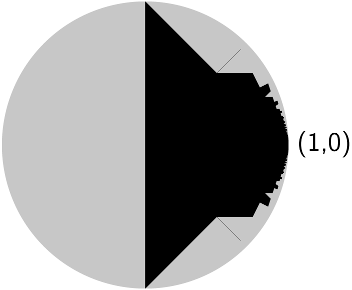


Figure: $\mathcal{G}_1^{(0)}$ in $\mathcal{G}_1$

Interested in $\mathcal{D}_d^{(0)}$ and $\mathcal{G}_d^{(0)}$

Why?

Motivation

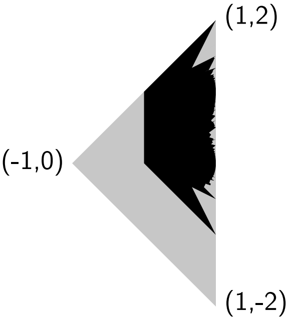


Figure: $\mathcal{D}_2^{(0)}$ in $\mathcal{D}_2$

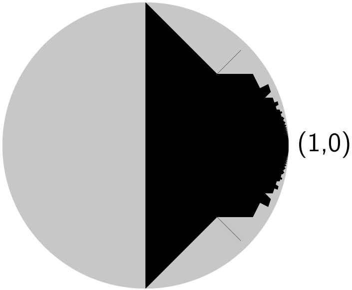


Figure: $\mathcal{G}_1^{(0)}$ in $\mathcal{G}_1$

Interested in $\mathcal{D}_d^{(0)}$ and $\mathcal{G}_d^{(0)}$

Why?

Relation between SRS, β-Expansions, and Canonical Number Systems

Let $\beta > 1$ be a non-integral, real number.

Let $\beta > 1$ be a non-integral, real number.

Then $\mathcal{A} := \{0, 1, \ldots, \lfloor \beta \rfloor\}$ is called the set of digits,

Let $\beta > 1$ be a non-integral, real number.

Then $\mathcal{A} := \{0, 1, \ldots, \lfloor\beta\rfloor\}$ is called the set of digits,
as every $\gamma \in [0, \infty)$ can be represented uniquely in the form

$\gamma = a_m\beta^m + a_{m-1}\beta^{m-1} + \cdots$
(greedy expansion of γ with respect to β)

with $m \in \mathbb{Z}$ and $a_i \in \mathcal{A}$, such that

$$0 \leq \gamma - \sum_{i=k}^{m} a_i\beta^i < \beta^k$$

holds for all $k \leq m$.

Motivation - Relation to β-Expansions

For $\gamma \in [0, 1)$ the greedy expansion can be given by the β-transformation

$T_\beta(\gamma) = \beta\gamma - \lfloor \beta\gamma \rfloor$

For $\gamma \in [0,1)$ the greedy expansion can be given by the β-transformation

$$T_\beta(\gamma) = \beta\gamma - \lfloor \beta\gamma \rfloor$$

$$\gamma = \lfloor \beta\gamma \rfloor \beta^{-1} + T_\beta(\gamma)\beta^{-1}$$

For $\gamma \in [0, 1)$ the greedy expansion can be given by the β-transformation

$T_\beta(\gamma) = \beta\gamma - \lfloor \beta\gamma \rfloor$

$\gamma = \lfloor \beta\gamma \rfloor \beta^{-1} + T_\beta(\gamma)\beta^{-1} = \lfloor \beta\gamma \rfloor \beta^{-1} + \lfloor \beta T_\beta(\gamma) \rfloor \beta^{-2} + T_\beta^2(\gamma)\beta^{-2}$

Motivation - Relation to β-Expansions

For $\gamma \in [0,1)$ the greedy expansion can be given by the β-transformation

$$T_\beta(\gamma) = \beta\gamma - \lfloor \beta\gamma \rfloor$$

$$\begin{aligned}\gamma &= \lfloor \beta\gamma \rfloor \beta^{-1} + T_\beta(\gamma)\beta^{-1} = \lfloor \beta\gamma \rfloor \beta^{-1} + \lfloor \beta T_\beta(\gamma) \rfloor \beta^{-2} + T_\beta^2(\gamma)\beta^{-2} \\ &= \sum_{i=1}^{k} \underbrace{\lfloor \beta T_\beta^{i-1}(\gamma) \rfloor}_{\text{digits}} \beta^{-i} + T_\beta^k(\gamma)\beta^{-k}\end{aligned}$$

Motivation - Relation to β-Expansions

For $\gamma \in [0,1)$ the greedy expansion can be given by the β-transformation

$$T_\beta(\gamma) = \beta\gamma - \lfloor \beta\gamma \rfloor$$

$$\begin{aligned}\gamma &= \lfloor \beta\gamma \rfloor \beta^{-1} + T_\beta(\gamma)\beta^{-1} = \lfloor \beta\gamma \rfloor \beta^{-1} + \lfloor \beta T_\beta(\gamma) \rfloor \beta^{-2} + T_\beta^2(\gamma)\beta^{-2} \\ &= \sum_{i=1}^{k} \underbrace{\lfloor \beta T_\beta^{i-1}(\gamma) \rfloor}_{\text{digits}} \beta^{-i} + T_\beta^k(\gamma)\beta^{-k}\end{aligned}$$

Example:

$\beta = \varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots \; (\Rightarrow \mathcal{A} = \{0,1\})$
$\gamma = \frac{5}{\varphi} - \frac{11}{\varphi^3} = 0.49342219125\ldots$

Motivation - Relation to β-Expansions

For $\gamma \in [0,1)$ the greedy expansion can be given by the β-transformation

$T_\beta(\gamma) = \beta\gamma - \lfloor \beta\gamma \rfloor$

$$\gamma = \lfloor \beta\gamma \rfloor \beta^{-1} + T_\beta(\gamma)\beta^{-1} = \lfloor \beta\gamma \rfloor \beta^{-1} + \lfloor \beta T_\beta(\gamma) \rfloor \beta^{-2} + T_\beta^2(\gamma)\beta^{-2}$$
$$= \sum_{i=1}^{k} \underbrace{\lfloor \beta T_\beta^{i-1}(\gamma) \rfloor}_{\text{digits}} \beta^{-i} + T_\beta^k(\gamma)\beta^{-k}$$

Example:

$\beta = \varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots \; (\Rightarrow \mathcal{A} = \{0,1\})$
$\gamma = \frac{5}{\varphi} - \frac{11}{\varphi^3} = 0.49342219125\ldots$

$0.493\ldots \; (0)$

Motivation - Relation to β-Expansions

For $\gamma \in [0,1)$ the greedy expansion can be given by the β-transformation

$$T_\beta(\gamma) = \beta\gamma - \lfloor \beta\gamma \rfloor$$

$$\gamma = \lfloor \beta\gamma \rfloor \beta^{-1} + T_\beta(\gamma)\beta^{-1} = \lfloor \beta\gamma \rfloor \beta^{-1} + \lfloor \beta T_\beta(\gamma) \rfloor \beta^{-2} + T_\beta^2(\gamma)\beta^{-2}$$
$$= \sum_{i=1}^{k} \underbrace{\lfloor \beta T_\beta^{i-1}(\gamma) \rfloor}_{\text{digits}} \beta^{-i} + T_\beta^k(\gamma)\beta^{-k}$$

Example:

$$\beta = \varphi = \tfrac{1+\sqrt{5}}{2} = 1.6180339887\ldots \ (\Rightarrow \mathcal{A} = \{0,1\})$$
$$\gamma = \tfrac{5}{\varphi} - \tfrac{11}{\varphi^3} = 0.49342219125\ldots$$

$$0.493\ldots\ (0) \xrightarrow{T_\beta} 0.798\ldots\ (1)$$

Motivation - Relation to β-Expansions

For $\gamma \in [0,1)$ the greedy expansion can be given by the β-transformation

$$T_\beta(\gamma) = \beta\gamma - \lfloor \beta\gamma \rfloor$$

$$\begin{aligned}\gamma &= \lfloor \beta\gamma \rfloor \beta^{-1} + T_\beta(\gamma)\beta^{-1} = \lfloor \beta\gamma \rfloor \beta^{-1} + \lfloor \beta T_\beta(\gamma) \rfloor \beta^{-2} + T_\beta^2(\gamma)\beta^{-2} \\ &= \sum_{i=1}^{k} \underbrace{\lfloor \beta T_\beta^{i-1}(\gamma) \rfloor}_{\text{digits}} \beta^{-i} + T_\beta^k(\gamma)\beta^{-k}\end{aligned}$$

Example:

$$\beta = \varphi = \tfrac{1+\sqrt{5}}{2} = 1.6180339887\ldots \; (\Rightarrow \mathcal{A} = \{0,1\})$$
$$\gamma = \tfrac{5}{\varphi} - \tfrac{11}{\varphi^3} = 0.49342219125\ldots$$

$$0.493\ldots\;(0) \xrightarrow{T_\beta} 0.798\ldots\;(1) \xrightarrow{T_\beta} 0.291\ldots\;(0)$$

Motivation - Relation to β-Expansions

For $\gamma \in [0,1)$ the greedy expansion can be given by the β-transformation

$$T_\beta(\gamma) = \beta\gamma - \lfloor \beta\gamma \rfloor$$

$$\begin{aligned}\gamma &= \lfloor \beta\gamma \rfloor \beta^{-1} + T_\beta(\gamma)\beta^{-1} = \lfloor \beta\gamma \rfloor \beta^{-1} + \lfloor \beta T_\beta(\gamma) \rfloor \beta^{-2} + T_\beta^2(\gamma)\beta^{-2} \\ &= \sum_{i=1}^{k} \underbrace{\lfloor \beta T_\beta^{i-1}(\gamma) \rfloor}_{\text{digits}} \beta^{-i} + T_\beta^k(\gamma)\beta^{-k}\end{aligned}$$

Example:

$\beta = \varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots \ (\Rightarrow \mathcal{A} = \{0,1\})$
$\gamma = \frac{5}{\varphi} - \frac{11}{\varphi^3} = 0.49342219125\ldots$

$$0.493\ldots\ (0) \xrightarrow{T_\beta} 0.798\ldots\ (1) \xrightarrow{T_\beta} 0.291\ldots\ (0) \xrightarrow{T_\beta} 0.472\ldots\ (0)$$

Motivation - Relation to β-Expansions

For $\gamma \in [0,1)$ the greedy expansion can be given by the β-transformation

$$T_\beta(\gamma) = \beta\gamma - \lfloor \beta\gamma \rfloor$$

$$\begin{aligned}\gamma &= \lfloor \beta\gamma \rfloor \beta^{-1} + T_\beta(\gamma)\beta^{-1} = \lfloor \beta\gamma \rfloor \beta^{-1} + \lfloor \beta T_\beta(\gamma) \rfloor \beta^{-2} + T_\beta^2(\gamma)\beta^{-2} \\ &= \sum_{i=1}^{k} \underbrace{\lfloor \beta T_\beta^{i-1}(\gamma) \rfloor}_{\text{digits}} \beta^{-i} + T_\beta^k(\gamma)\beta^{-k}\end{aligned}$$

Example:

$\beta = \varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots \; (\Rightarrow \mathcal{A} = \{0,1\})$
$\gamma = \frac{5}{\varphi} - \frac{11}{\varphi^3} = 0.49342219125\ldots$

$0.493\ldots\ (0) \xrightarrow{T_\beta} 0.798\ldots\ (1) \xrightarrow{T_\beta} 0.291\ldots\ (0) \xrightarrow{T_\beta} 0.472\ldots\ (0) \xrightarrow{T_\beta}$
$0.763\ldots\ (1)$

For $\gamma \in [0,1)$ the greedy expansion can be given by the β-transformation

$$T_\beta(\gamma) = \beta\gamma - \lfloor \beta\gamma \rfloor$$

$$\begin{aligned}\gamma &= \lfloor \beta\gamma \rfloor \beta^{-1} + T_\beta(\gamma)\beta^{-1} = \lfloor \beta\gamma \rfloor \beta^{-1} + \lfloor \beta T_\beta(\gamma) \rfloor \beta^{-2} + T_\beta^2(\gamma)\beta^{-2} \\ &= \sum_{i=1}^{k} \underbrace{\lfloor \beta T_\beta^{i-1}(\gamma) \rfloor}_{\text{digits}} \beta^{-i} + T_\beta^k(\gamma)\beta^{-k}\end{aligned}$$

Example:

$\beta = \varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots \; (\Rightarrow \mathcal{A} = \{0,1\})$
$\gamma = \frac{5}{\varphi} - \frac{11}{\varphi^3} = 0.49342219125\ldots$

$0.493\ldots\;(0) \xrightarrow{T_\beta} 0.798\ldots\;(1) \xrightarrow{T_\beta} 0.291\ldots\;(0) \xrightarrow{T_\beta} 0.472\ldots\;(0) \xrightarrow{T_\beta}$
$0.763\ldots\;(1) \xrightarrow{T_\beta} 0.236\ldots\;(0)$

For $\gamma \in [0,1)$ the greedy expansion can be given by the β-transformation

$$T_\beta(\gamma) = \beta\gamma - \lfloor \beta\gamma \rfloor$$

$$\gamma = \lfloor \beta\gamma \rfloor \beta^{-1} + T_\beta(\gamma)\beta^{-1} = \lfloor \beta\gamma \rfloor \beta^{-1} + \lfloor \beta T_\beta(\gamma) \rfloor \beta^{-2} + T_\beta^2(\gamma)\beta^{-2}$$
$$= \sum_{i=1}^{k} \underbrace{\lfloor \beta T_\beta^{i-1}(\gamma) \rfloor}_{\text{digits}} \beta^{-i} + T_\beta^k(\gamma)\beta^{-k}$$

Example:

$$\beta = \varphi = \tfrac{1+\sqrt{5}}{2} = 1.6180339887\ldots \ (\Rightarrow \mathcal{A} = \{0,1\})$$
$$\gamma = \tfrac{5}{\varphi} - \tfrac{11}{\varphi^3} = 0.49342219125\ldots$$

$$0.493\ldots\ (0) \xrightarrow{T_\beta} 0.798\ldots\ (1) \xrightarrow{T_\beta} 0.291\ldots\ (0) \xrightarrow{T_\beta} 0.472\ldots\ (0) \xrightarrow{T_\beta}$$
$$0.763\ldots\ (1) \xrightarrow{T_\beta} 0.236\ldots\ (0) \xrightarrow{T_\beta} 0.381\ldots\ (0)$$

For $\gamma \in [0,1)$ the greedy expansion can be given by the β-transformation

$$T_\beta(\gamma) = \beta\gamma - \lfloor \beta\gamma \rfloor$$

$$\begin{aligned}\gamma &= \lfloor \beta\gamma \rfloor \beta^{-1} + T_\beta(\gamma)\beta^{-1} = \lfloor \beta\gamma \rfloor \beta^{-1} + \lfloor \beta T_\beta(\gamma) \rfloor \beta^{-2} + T_\beta^2(\gamma)\beta^{-2} \\ &= \sum_{i=1}^{k} \underbrace{\lfloor \beta T_\beta^{i-1}(\gamma) \rfloor}_{\text{digits}} \beta^{-i} + T_\beta^k(\gamma)\beta^{-k}\end{aligned}$$

Example:

$\beta = \varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots\ (\Rightarrow \mathcal{A} = \{0,1\})$
$\gamma = \frac{5}{\varphi} - \frac{11}{\varphi^3} = 0.49342219125\ldots$

$0.493\ldots\ (0) \xrightarrow{T_\beta} 0.798\ldots\ (1) \xrightarrow{T_\beta} 0.291\ldots\ (0) \xrightarrow{T_\beta} 0.472\ldots\ (0) \xrightarrow{T_\beta}$
$0.763\ldots\ (1) \xrightarrow{T_\beta} 0.236\ldots\ (0) \xrightarrow{T_\beta} 0.381\ldots\ (0) \xrightarrow{T_\beta} 0.618\ldots\ (1)$

For $\gamma \in [0,1)$ the greedy expansion can be given by the β-transformation

$$T_\beta(\gamma) = \beta\gamma - \lfloor \beta\gamma \rfloor$$

$$\begin{aligned} \gamma &= \lfloor \beta\gamma \rfloor \beta^{-1} + T_\beta(\gamma)\beta^{-1} = \lfloor \beta\gamma \rfloor \beta^{-1} + \lfloor \beta T_\beta(\gamma) \rfloor \beta^{-2} + T_\beta^2(\gamma)\beta^{-2} \\ &= \sum_{i=1}^{k} \underbrace{\lfloor \beta T_\beta^{i-1}(\gamma) \rfloor}_{\text{digits}} \beta^{-i} + T_\beta^k(\gamma)\beta^{-k} \end{aligned}$$

Example:

$$\beta = \varphi = \tfrac{1+\sqrt{5}}{2} = 1.6180339887\ldots \; (\Rightarrow \mathcal{A} = \{0,1\})$$
$$\gamma = \tfrac{5}{\varphi} - \tfrac{11}{\varphi^3} = 0.49342219125\ldots$$

$$0.493\ldots\,(0) \xrightarrow{T_\beta} 0.798\ldots\,(1) \xrightarrow{T_\beta} 0.291\ldots\,(0) \xrightarrow{T_\beta} 0.472\ldots\,(0) \xrightarrow{T_\beta}$$
$$0.763\ldots\,(1) \xrightarrow{T_\beta} 0.236\ldots\,(0) \xrightarrow{T_\beta} 0.381\ldots\,(0) \xrightarrow{T_\beta} 0.618\ldots\,(1) \xrightarrow{T_\beta}$$
$$0\ldots\,(0)$$

For $\gamma \in [0,1)$ the greedy expansion can be given by the β-transformation

$$T_\beta(\gamma) = \beta\gamma - \lfloor \beta\gamma \rfloor$$

$$\begin{aligned}\gamma &= \lfloor \beta\gamma \rfloor \beta^{-1} + T_\beta(\gamma)\beta^{-1} = \lfloor \beta\gamma \rfloor \beta^{-1} + \lfloor \beta T_\beta(\gamma) \rfloor \beta^{-2} + T_\beta^2(\gamma)\beta^{-2} \\ &= \sum_{i=1}^{k} \underbrace{\lfloor \beta T_\beta^{i-1}(\gamma) \rfloor}_{\text{digits}} \beta^{-i} + T_\beta^k(\gamma)\beta^{-k}\end{aligned}$$

Example:

$$\beta = \varphi = \tfrac{1+\sqrt{5}}{2} = 1.6180339887\ldots \; (\Rightarrow \mathcal{A} = \{0,1\})$$
$$\gamma = \tfrac{5}{\varphi} - \tfrac{11}{\varphi^3} = 0.49342219125\ldots$$

$$\begin{aligned}&0.493\ldots\,(0) \xrightarrow{T_\beta} 0.798\ldots\,(1) \xrightarrow{T_\beta} 0.291\ldots\,(0) \xrightarrow{T_\beta} 0.472\ldots\,(0) \xrightarrow{T_\beta} \\ &0.763\ldots\,(1) \xrightarrow{T_\beta} 0.236\ldots\,(0) \xrightarrow{T_\beta} 0.381\ldots\,(0) \xrightarrow{T_\beta} 0.618\ldots\,(1) \xrightarrow{T_\beta} \\ &0\ldots\,(0) \qquad \xrightarrow{T_\beta} 0\ldots\,(0)\ \ldots\end{aligned}$$

For $\gamma \in [0,1)$ the greedy expansion can be given by the β-transformation

$$T_\beta(\gamma) = \beta\gamma - \lfloor \beta\gamma \rfloor$$

$$\begin{aligned}\gamma &= \lfloor \beta\gamma \rfloor \beta^{-1} + T_\beta(\gamma)\beta^{-1} = \lfloor \beta\gamma \rfloor \beta^{-1} + \lfloor \beta T_\beta(\gamma) \rfloor \beta^{-2} + T_\beta^2(\gamma)\beta^{-2} \\ &= \sum_{i=1}^{k} \underbrace{\lfloor \beta T_\beta^{i-1}(\gamma) \rfloor}_{\text{digits}} \beta^{-i} + T_\beta^k(\gamma)\beta^{-k}\end{aligned}$$

Example:

$$\beta = \varphi = \tfrac{1+\sqrt{5}}{2} = 1.6180339887\ldots \ (\Rightarrow \mathcal{A} = \{0,1\})$$
$$\gamma = \tfrac{5}{\varphi} - \tfrac{11}{\varphi^3} = 0.49342219125\ldots$$

$$\begin{array}{l} 0.493\ldots\ (0) \xrightarrow{T_\beta} 0.798\ldots\ (1) \xrightarrow{T_\beta} 0.291\ldots\ (0) \xrightarrow{T_\beta} 0.472\ldots\ (0) \xrightarrow{T_\beta} \\ 0.763\ldots\ (1) \xrightarrow{T_\beta} 0.236\ldots\ (0) \xrightarrow{T_\beta} 0.381\ldots\ (0) \xrightarrow{T_\beta} 0.618\ldots\ (1) \xrightarrow{T_\beta} \\ 0\ldots\ (0) \quad \xrightarrow{T_\beta} 0\ldots\ (0)\ \ldots \end{array}$$

$$\gamma = 0 \cdot \tfrac{1}{\beta} + 1 \cdot \tfrac{1}{\beta^2} + 0 \cdot \tfrac{1}{\beta^3} + 0 \cdot \tfrac{1}{\beta^4} + 1 \cdot \tfrac{1}{\beta^5} + 0 \cdot \tfrac{1}{\beta^6} + 0 \cdot \tfrac{1}{\beta^7} + 1 \cdot \tfrac{1}{\beta^8}$$

Let *Fin*(β) be the set of all $\gamma \in [0, 1)$ having finite greedy expansion with respect to β.

Motivation - Relation to β-Expansions

Let $\mathit{Fin}(\beta)$ be the set of all $\gamma \in [0, 1)$ having finite greedy expansion with respect to β.

Then $\mathit{Fin}(\beta) \subseteq \mathbb{Z}[\frac{1}{\beta}] \cap [0, 1)$

Motivation - Relation to β-Expansions

Let $Fin(\beta)$ be the set of all $\gamma \in [0,1)$ having finite greedy expansion with respect to β.

Then $Fin(\beta) \subseteq \mathbb{Z}[\frac{1}{\beta}] \cap [0,1)$

If $Fin(\beta) = \mathbb{Z}[\frac{1}{\beta}] \cap [0,1)$ then β is said to have property (F).

Let $Fin(\beta)$ be the set of all $\gamma \in [0,1)$ having finite greedy expansion with respect to β.

Then $Fin(\beta) \subseteq \mathbb{Z}[\frac{1}{\beta}] \cap [0,1)$

If $Fin(\beta) = \mathbb{Z}[\frac{1}{\beta}] \cap [0,1)$ then β is said to have property (F).

In that case β is an algebraic integer (furthermore a Pisot number) and therefore has a minimal polynomial

$$X^d + a_{d-1}X^{d-1} + \cdots + a_1 X + a_0 \in \mathbb{Z}[X]$$

Motivation - Relation to β-Expansions

Let $Fin(\beta)$ be the set of all $\gamma \in [0,1)$ having finite greedy expansion with respect to β.

Then $Fin(\beta) \subseteq \mathbb{Z}[\frac{1}{\beta}] \cap [0,1)$

If $Fin(\beta) = \mathbb{Z}[\frac{1}{\beta}] \cap [0,1)$ then β is said to have property (F).

In that case β is an algebraic integer (furthermore a Pisot number) and therefore has a minimal polynomial

$$X^d + a_{d-1}X^{d-1} + \cdots + a_1 X + a_0 \in \mathbb{Z}[X]$$

which can be written as

$$(X - \beta)(X^{d-1} + r_{d-2}X^{d-2} + \cdots + r_1 X + r_0)$$

Motivation - Relation to β-Expansions

Let $Fin(\beta)$ be the set of all $\gamma \in [0,1)$ having finite greedy expansion with respect to β.

Then $Fin(\beta) \subseteq \mathbb{Z}[\frac{1}{\beta}] \cap [0,1)$

If $Fin(\beta) = \mathbb{Z}[\frac{1}{\beta}] \cap [0,1)$ then β is said to have property (F).

In that case β is an algebraic integer (furthermore a Pisot number) and therefore has a minimal polynomial

$$X^d + a_{d-1}X^{d-1} + \cdots + a_1 X + a_0 \in \mathbb{Z}[X]$$

which can be written as

$$(X - \beta)(X^{d-1} + r_{d-2}X^{d-2} + \cdots + r_1 X + r_0)$$

Then β has property (F) $\iff$ $(r_0, \ldots, r_{d-2}) \in \mathcal{D}_{d-1}^{(0)}$ (AKIYAMA et al. 2005)

A similar relation can be shown for CNS:

Motivation - Relation to Canonical Number Systems

A similar relation can be shown for CNS:

Let $P(X) = X^d + p_{d-1}X^{d-1} + \cdots + p_1 X + p_0 \in \mathbb{Z}[X]$

A similar relation can be shown for CNS:

Let $P(X) = X^d + p_{d-1}X^{d-1} + \cdots + p_1 X + p_0 \in \mathbb{Z}[X]$

Then P is a CNS polynomial $\iff (\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \ldots, \frac{p_2}{p_0}, \frac{p_1}{p_0}) \in \mathcal{D}_d^{(0)}$
(AKIYAMA et al. 2005)

For $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$ let

$$R_{\mathbf{r}} := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -r_1 & -r_2 & \cdots & \cdots & -r_d \end{pmatrix}$$

– the companion matrix of $\chi_{\mathbf{r}}(X) = X^d + r_d X^{d-1} + \cdots + r_2 X + r_1$.

For $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$ let

$$R_{\mathbf{r}} := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -r_1 & -r_2 & \cdots & \cdots & -r_d \end{pmatrix}$$

– the companion matrix of $\chi_{\mathbf{r}}(X) = X^d + r_d X^{d-1} + \cdots + r_2 X + r_1$.

Then $\tau_{\mathbf{r}}(\mathbf{x}) = R_{\mathbf{r}}\mathbf{x} + \mathbf{v}_{\mathbf{x}}$ where $\mathbf{v}_{\mathbf{x}} = (0, \ldots, 0, \{\mathbf{rx}\})$.

For $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$ let

$$R_{\mathbf{r}} := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -r_1 & -r_2 & \cdots & \cdots & -r_d \end{pmatrix}$$

– the companion matrix of $\chi_{\mathbf{r}}(X) = X^d + r_d X^{d-1} + \cdots + r_2 X + r_1$.

Then $\tau_{\mathbf{r}}(\mathbf{x}) = R_{\mathbf{r}}\mathbf{x} + \mathbf{v}_{\mathbf{x}}$ where $\mathbf{v}_{\mathbf{x}} = (0, \ldots, 0, \{\mathbf{rx}\})$.

Let $\rho(M)$ denote the spectral radius of a matrix.
(i.e. the maximum absolute value of eigenvalues)

For $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$ let

$$R_{\mathbf{r}} := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -r_1 & -r_2 & \cdots & \cdots & -r_d \end{pmatrix}$$

– the companion matrix of $\chi_{\mathbf{r}}(X) = X^d + r_d X^{d-1} + \cdots + r_2 X + r_1$.

Then $\tau_{\mathbf{r}}(\mathbf{x}) = R_{\mathbf{r}}\mathbf{x} + \mathbf{v_x}$ where $\mathbf{v_x} = (0, \ldots, 0, \{\mathbf{rx}\})$.

Let $\rho(M)$ denote the spectral radius of a matrix.
(i.e. the maximum absolute value of eigenvalues)

Then • $\mathcal{D}_d \quad \subseteq \{\mathbf{r} \in \mathbb{R}^d \mid \rho(R_{\mathbf{r}}) \leq 1\}$

For $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{R}^d$ let

$$R_{\mathbf{r}} := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -r_1 & -r_2 & \cdots & \cdots & -r_d \end{pmatrix}$$

– the companion matrix of $\chi_{\mathbf{r}}(X) = X^d + r_d X^{d-1} + \cdots + r_2 X + r_1$.

Then $\tau_{\mathbf{r}}(\mathbf{x}) = R_{\mathbf{r}}\mathbf{x} + \mathbf{v}_{\mathbf{x}}$ where $\mathbf{v}_{\mathbf{x}} = (0, \dots, 0, \{\mathbf{rx}\})$.

Let $\rho(M)$ denote the spectral radius of a matrix.
(i.e. the maximum absolute value of eigenvalues)

Then
- $\mathcal{D}_d \subseteq \{\mathbf{r} \in \mathbb{R}^d \mid \rho(R_{\mathbf{r}}) \leq 1\}$
- $\{\mathbf{r} \in \mathbb{R}^d \mid \rho(R_{\mathbf{r}}) < 1\} \subseteq \mathcal{D}_d$

For $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$ let

$$R_{\mathbf{r}} := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -r_1 & -r_2 & \cdots & \cdots & -r_d \end{pmatrix}$$

– the companion matrix of $\chi_{\mathbf{r}}(X) = X^d + r_d X^{d-1} + \cdots + r_2 X + r_1$.

Then $\tau_{\mathbf{r}}(\mathbf{x}) = R_{\mathbf{r}}\mathbf{x} + \mathbf{v}_{\mathbf{x}}$ where $\mathbf{v}_{\mathbf{x}} = (0, \ldots, 0, \{\mathbf{rx}\})$.

Let $\rho(M)$ denote the spectral radius of a matrix.
(i.e. the maximum absolute value of eigenvalues)

Then
- $\mathcal{D}_d \subseteq \{\mathbf{r} \in \mathbb{R}^d \mid \rho(R_{\mathbf{r}}) \le 1\}$
- $\{\mathbf{r} \in \mathbb{R}^d \mid \rho(R_{\mathbf{r}}) < 1\} \subseteq \mathcal{D}_d$
- $\partial\mathcal{D}_d = \{\mathbf{r} \in \mathbb{R}^d \mid \rho(R_{\mathbf{r}}) = 1\}$

For $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$ let

$$R_{\mathbf{r}} := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -r_1 & -r_2 & \cdots & \cdots & -r_d \end{pmatrix}$$

– the companion matrix of $\chi_{\mathbf{r}}(X) = X^d + r_d X^{d-1} + \cdots + r_2 X + r_1$.

Then $\tau_{\mathbf{r}}(\mathbf{x}) = R_{\mathbf{r}}\mathbf{x} + \mathbf{v}_{\mathbf{x}}$ where $\mathbf{v}_{\mathbf{x}} = (0, \ldots, 0, \{\mathbf{r}\mathbf{x}\})$.

Let $\rho(M)$ denote the spectral radius of a matrix.
(i.e. the maximum absolute value of eigenvalues)

Then
- $\mathcal{D}_d \subseteq \{\mathbf{r} \in \mathbb{R}^d \mid \rho(R_{\mathbf{r}}) \leq 1\}$
- $\{\mathbf{r} \in \mathbb{R}^d \mid \rho(R_{\mathbf{r}}) < 1\} \subseteq \mathcal{D}_d$
- $\partial\mathcal{D}_d = \{\mathbf{r} \in \mathbb{R}^d \mid \rho(R_{\mathbf{r}}) = 1\}$

Equivalent statements are true for $\mathcal{G}_d$.

Characterization of $\mathcal{D}_d^{(0)}$ and $\mathcal{G}_d^{(0)}$– Two important concepts

Cutout polyhedra

For a tuple π of vectors in $\mathbb{Z}^d$ let $\mathcal{P}(\pi)$ denote the set of all $\mathbf{r} \in \mathbb{R}^d$ for which π is a period of $\tau_{\mathbf{r}}$.

$\pi = (\mathbf{x_1}, \ldots, \mathbf{x_n})$, $\tau_{\mathbf{r}}(\mathbf{x_1}) = \mathbf{x_2}$, $\ldots$,$\tau_{\mathbf{r}}(\mathbf{x_n}) = \mathbf{x_1}$

Characterization of $\mathcal{D}_d^{(0)}$ and $\mathcal{G}_d^{(0)}$– Two important concepts

Cutout polyhedra

For a tuple π of vectors in $\mathbb{Z}^d$ let $\mathcal{P}(\pi)$ denote the set of all $\mathbf{r} \in \mathbb{R}^d$ for which π is a period of $\tau_{\mathbf{r}}$.

$\pi = (\mathbf{x_1}, \ldots, \mathbf{x_n})$, $\tau_{\mathbf{r}}(\mathbf{x_1}) = \mathbf{x_2}$, ... ,$\tau_{\mathbf{r}}(\mathbf{x_n}) = \mathbf{x_1}$

Then

$\mathcal{P}(\pi)$ is a (possibly degenerate) convex polyhedron characterized by a finite set of linear inequalities

Characterization of $\mathcal{D}_d^{(0)}$ and $\mathcal{G}_d^{(0)}$– Two important concepts

Cutout polyhedra

For a tuple π of vectors in $\mathbb{Z}^d$ let $\mathcal{P}(\pi)$ denote the set of all $\mathbf{r} \in \mathbb{R}^d$ for which π is a period of $\tau_{\mathbf{r}}$.

$\pi = (\mathbf{x_1}, \ldots, \mathbf{x_n})$, $\tau_{\mathbf{r}}(\mathbf{x_1}) = \mathbf{x_2}$, $\ldots$,$\tau_{\mathbf{r}}(\mathbf{x_n}) = \mathbf{x_1}$

Then

$\mathcal{P}(\pi)$ is a (possibly degenerate) convex polyhedron characterized by a finite set of linear inequalities

$\mathcal{D}_d^{(0)} = \mathcal{D}_d \setminus \bigcup_{\pi \neq 0} \mathcal{P}(\pi)$ (BRUNOTTE 2001)

Characterization of $\mathcal{D}_d^{(0)}$ and $\mathcal{G}_d^{(0)}$– Two important concepts

Cutout polyhedra

For a tuple π of vectors in $\mathbb{Z}^d$ let $\mathcal{P}(\pi)$ denote the set of all $\mathbf{r} \in \mathbb{R}^d$ for which π is a period of $\tau_{\mathbf{r}}$.

$\pi = (\mathbf{x_1}, \ldots, \mathbf{x_n})$, $\tau_{\mathbf{r}}(\mathbf{x_1}) = \mathbf{x_2}$, $\ldots$,$\tau_{\mathbf{r}}(\mathbf{x_n}) = \mathbf{x_1}$

Then

$\mathcal{P}(\pi)$ is a (possibly degenerate) convex polyhedron characterized by a finite set of linear inequalities

$\mathcal{D}_d^{(0)} = \mathcal{D}_d \setminus \bigcup_{\pi \neq 0} \mathcal{P}(\pi)$ (BRUNOTTE 2001)

Equivalent statements are true for $\mathcal{G}_d$ and $\mathcal{G}_d^{(0)}$.

Characterization of $\mathcal{D}_d^{(0)}$ and $\mathcal{G}_d^{(0)}$ – Two important concepts

Sets of witnesses

A set $V \subseteq \mathbb{Z}^d$ is called a set of witnesses for $\mathbf{r} \in \mathbb{R}^d$ iff it is stable under $\tau_{\mathbf{r}}$ and $\tau_{\mathbf{r}}^\star := -\tau_{\mathbf{r}} \circ (-\mathrm{id}_{\mathbb{Z}^d})$ and contains a generating set of the group $(\mathbb{Z}^d, +)$ which is closed under taking inverses.

Characterization of $\mathcal{D}_d^{(0)}$ and $\mathcal{G}_d^{(0)}$ – Two important concepts

Sets of witnesses

A set $V \subseteq \mathbb{Z}^d$ is called a set of witnesses for $\mathbf{r} \in \mathbb{R}^d$ iff it is stable under $\tau_{\mathbf{r}}$ and $\tau_{\mathbf{r}}^{\star} := -\tau_{\mathbf{r}} \circ (-\mathrm{id}_{\mathbb{Z}^d})$ and contains a generating set of the group $(\mathbb{Z}^d, +)$ which is closed under taking inverses.

Every such set of witnesses has the decisive property:

$\mathbf{r} \in \mathcal{D}_d^{(0)} \Leftrightarrow \forall\, \mathbf{a} \in V : \exists\, n \in \mathbb{N} : \tau_{\mathbf{r}}^n(\mathbf{a}) = \mathbf{0}$

Characterization of $\mathcal{D}_d^{(0)}$ and $\mathcal{G}_d^{(0)}$ – Two important concepts

Sets of witnesses

A set $V \subseteq \mathbb{Z}^d$ is called a set of witnesses for $\mathbf{r} \in \mathbb{R}^d$ iff it is stable under $\tau_{\mathbf{r}}$ and $\tau_{\mathbf{r}}^\star := -\tau_{\mathbf{r}} \circ (-\mathrm{id}_{\mathbb{Z}^d})$ and contains a generating set of the group $(\mathbb{Z}^d, +)$ which is closed under taking inverses.

Every such set of witnesses has the decisive property:

$$\mathbf{r} \in \mathcal{D}_d^{(0)} \Leftrightarrow \forall\, \mathbf{a} \in V : \exists\, n \in \mathbb{N} : \tau_{\mathbf{r}}^n(\mathbf{a}) = \mathbf{0}$$

Find a finite set of witnesses iteratively for $\mathbf{r} \in \mathrm{int}\,(\mathcal{D}_d)$:

$$V_0 := \{(\pm 1, 0, \dots, 0), \dots, (0, \dots, 0, \pm 1)\}$$
$$\forall\, n \in \mathbb{N} : V_n := V_{n-1} \cup \tau_{\mathbf{r}}(V_{n-1}) \cup \tau_{\mathbf{r}}^\star(V_{n-1})$$
$$V_{\mathbf{r}} := \bigcup_{n \in \mathbb{N}_0} V_n$$

(BRUNOTTE 2001)

Characterization of $\mathcal{D}_d^{(0)}$ and $\mathcal{G}_d^{(0)}$ – Two important concepts

Sets of witnesses

A set $V \subseteq \mathbb{Z}^d$ is called a set of witnesses for $\mathbf{r} \in \mathbb{R}^d$ iff it is stable under $\tau_{\mathbf{r}}$ and $\tau_{\mathbf{r}}^{\star} := -\tau_{\mathbf{r}} \circ (-\mathrm{id}_{\mathbb{Z}^d})$ and contains a generating set of the group $(\mathbb{Z}^d, +)$ which is closed under taking inverses.

Every such set of witnesses has the decisive property:

$$\mathbf{r} \in \mathcal{D}_d^{(0)} \Leftrightarrow \forall\, \mathbf{a} \in V : \exists\, n \in \mathbb{N} : \tau_{\mathbf{r}}^n(\mathbf{a}) = \mathbf{0}$$

Find a finite set of witnesses iteratively for $\mathbf{r} \in \mathrm{int}\,(\mathcal{D}_d)$:

$$\begin{aligned}
V_0 &:= \{(\pm 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, \pm 1)\} \\
\forall\, n \in \mathbb{N} : V_n &:= V_{n-1} \cup \tau_{\mathbf{r}}(V_{n-1}) \cup \tau_{\mathbf{r}}^{\star}(V_{n-1}) \\
V_{\mathbf{r}} &:= \bigcup_{n \in \mathbb{N}_0} V_n
\end{aligned}$$

(BRUNOTTE 2001)

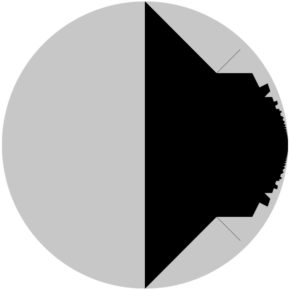
Equivalent statements are true for $\mathcal{G}_d$ and $\mathcal{G}_d^{(0)}$.

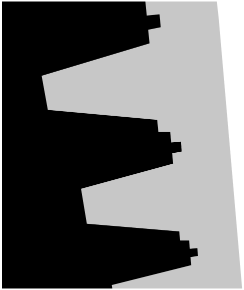
Characterization of $\mathcal{D}_d^{(0)}$ and $\mathcal{G}_d^{(0)}$ – Two important concepts

Example: V for $r = \frac{9}{10} + \mathrm{i}\frac{6}{17}$

Characterization of $\mathcal{D}_d^{(0)}$ and $\mathcal{G}_d^{(0)}$ – Two important concepts

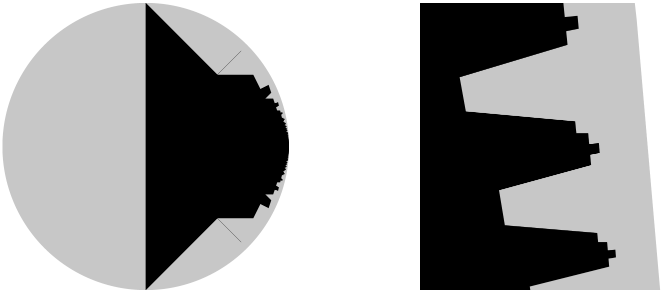
Example: V for $r = \frac{9}{10} + \mathrm{i}\frac{6}{17}$





$\mathcal{G}_1^{(0)}$ is contained in the closed right half of the closed unit disk, and is symmetric with respect to the real axis (BRUNOTTE et al. 2011).

Characterization of $\mathcal{G}_1^{(0)}$ (Pethő's Loudspeaker)



$\mathcal{G}_1^{(0)}$ is contained in the closed right half of the closed unit disk, and is symmetric with respect to the real axis (BRUNOTTE et al. 2011).

Conjecture by M.W.:

$\mathcal{G}_1^{(0)} = \mathcal{G}_C$ where $\mathcal{G}_C$ is a (neither open nor closed) polygon given by ten infinite sequences of points in $\mathbb{C}$ (and their complex conjugates).

Characterization of $\mathcal{G}_1^{(0)}$ (Pethő's Loudspeaker)

Ten sequences:

$$z_1(n) = 1 + \frac{-2+\mathrm{i}n}{n^2-2}$$

$$z_2(n) = 1 + \frac{-1+\mathrm{i}(n-1)}{n^2-n-1}$$

$$z_3(n) = 1 + \frac{-1+\mathrm{i}(n-1)}{n^2-n}$$

$$z_4(n) = 1 + \frac{-1+\mathrm{i}n}{n^2}$$

$$z_5(n) = 1 + \frac{-1+\mathrm{i}n}{n^2+1}$$

$$z_6(n) = 1 + \frac{-1+\mathrm{i}(n+1)}{n^2+n+1}$$

$$z_7(n) = 1 + \frac{-1+\mathrm{i}(n+1)}{n^2+n+2}$$

$$z_8(n) = 1 + \frac{-1+\mathrm{i}n}{n^2+2}$$

$$z_9(n) = 1 + \frac{-1+\mathrm{i}n}{n^2+3}$$

$$z_{10}(n) = 1 + \frac{-2+\mathrm{i}(n+1)}{n^2+n+6}$$

Already proven: $\mathcal{G}_1^{(0)} \subseteq \mathcal{G}_C$

Result achieved by identification of 20 families of cutout polygons

Characterization of $\mathcal{G}_1^{(0)}$ (Pethő's Loudspeaker)

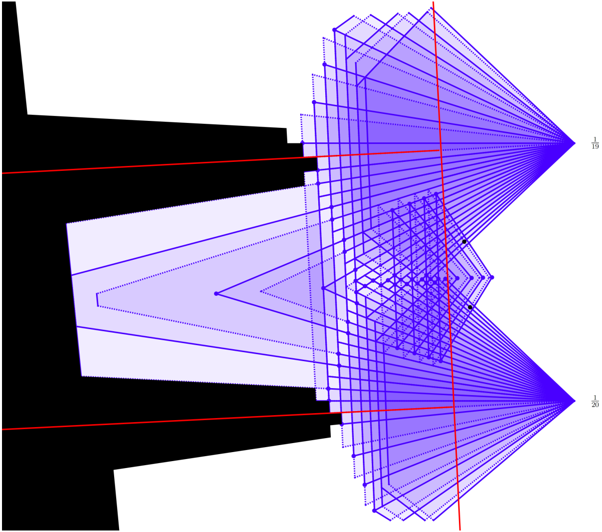


Figure: 20 families of cutout polygons

Characterization of $\mathcal{G}_1^{(0)}$ (Pethő's Loudspeaker)

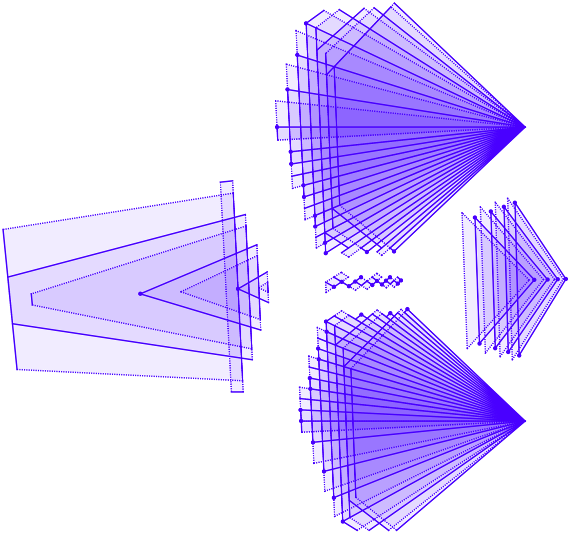


Figure: 20 families of cutout polygons

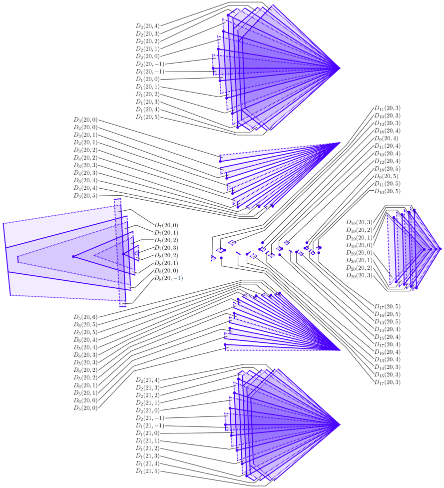


Figure: 20 families of cutout polygons

Characterization of $\mathcal{G}_1^{(0)}$ (Pethő's Loudspeaker)

Other inclusion: Settled inside $\left\{\mathbf{r} \in \mathbb{C} \mid |\mathbf{r}| \leq \frac{1023}{1024}\right\}$ by a new algorithm

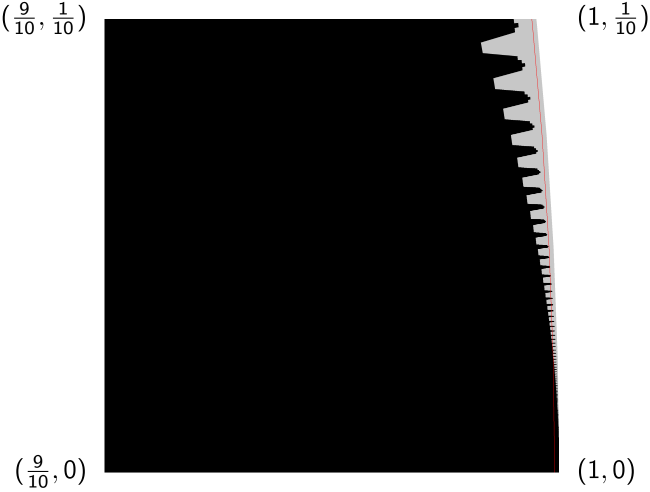


Figure: Region settled algorithmically

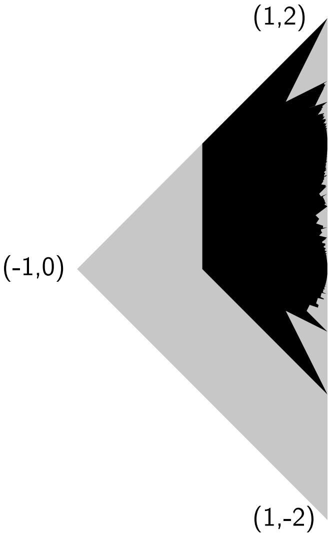
Hope for full characterization of $\mathcal{G}_1^{(0)}$
by thorough investigation of orbits of $\gamma_{\mathbf{r}}$!

Characterization of $\mathcal{D}_d^{(0)}$ – Previous results

- $\mathcal{D}_1 = [-1, 1]$, $\mathcal{D}_1^{(0)} = [0, 1)$

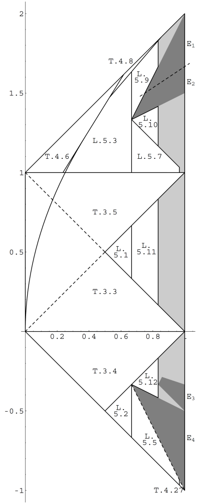
Characterization of $\mathcal{D}_d^{(0)}$ – Previous results

- $\mathcal{D}_1 = [-1, 1]$, $\mathcal{D}_1^{(0)} = [0, 1)$
- $\mathcal{D}_2 \subseteq \{(x, y) \in \mathbb{R}^2 \mid x \geq |y| - 1 \wedge x \leq 1\}$



Characterization of $\mathcal{D}_d^{(0)}$ – Previous results

- $\mathcal{D}_1 = [-1, 1]$, $\mathcal{D}_1^{(0)} = [0, 1)$
- $\mathcal{D}_2 \subseteq \{(x, y) \in \mathbb{R}^2 \mid x \geq |y| - 1 \wedge x \leq 1\}$
- Several regions of $\mathcal{D}_2^{(0)}$ have been characterized by AKIYAMA et al. in 2005

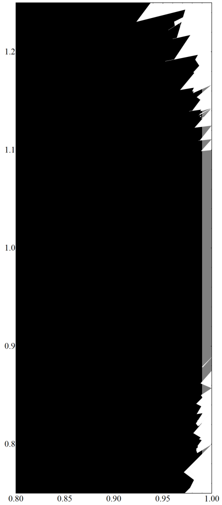


Characterization of $\mathcal{D}_d^{(0)}$ – Previous results

- $\mathcal{D}_1 = [-1, 1]$, $\mathcal{D}_1^{(0)} = [0, 1)$
- $\mathcal{D}_2 \subseteq \{(x, y) \in \mathbb{R}^2 \mid x \geq |y| - 1 \wedge x \leq 1\}$
- Several regions of $\mathcal{D}_2^{(0)}$ have been characterized by AKIYAMA et al. in 2005
- Adaptation of the concept of sets of witnesses leads to an algorithm due to Brunotte

- $\mathcal{D}_1 = [-1, 1]$, $\mathcal{D}_1^{(0)} = [0, 1)$
- $\mathcal{D}_2 \subseteq \{(x, y) \in \mathbb{R}^2 \mid x \geq |y| - 1 \wedge x \leq 1\}$
- Several regions of $\mathcal{D}_2^{(0)}$ have been characterized by AKIYAMA et al. in 2005
- Adaptation of the concept of sets of witnesses leads to an algorithm due to Brunotte

  Applied by SURER 2008 to characterize $\mathcal{D}_2^{(0)} \cap \left\{(x, y) \in \mathbb{R}^2 \mid x \leq L\right\}$ where $L = \frac{99}{100}$



Characterization of $\mathcal{D}_d^{(0)}$ – New results

Theorem (M.W.):

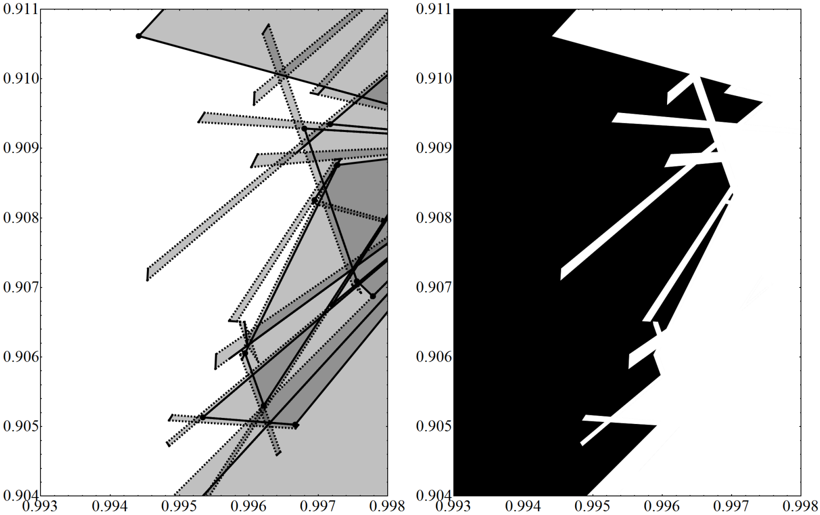
- $\mathcal{D}_2^{(0)}$ has at least 22 connected components
- The largest connected component of $\mathcal{D}_2^{(0)}$ has at least 3 holes

Theorem (M.W.):

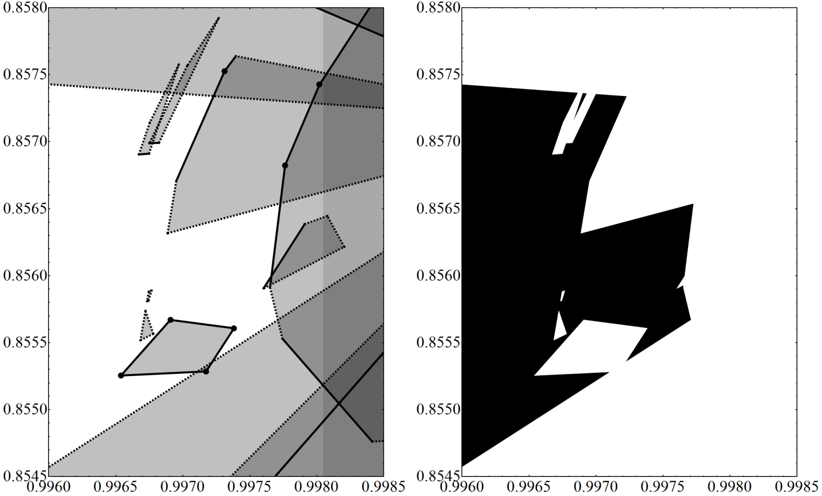
- $\mathcal{D}_2^{(0)}$ has at least 22 connected components
- The largest connected component of $\mathcal{D}_2^{(0)}$ has at least 3 holes

Result achieved by a new algorithm which has been used to characterize $\mathcal{D}_2^{(0)} \cap \{(x, y) \in \mathbb{R}^2 \mid x \leq L\}$ where $L = \frac{511}{512}$.

Characterization of $\mathcal{D}_d^{(0)}$ – New results



Characterization of $\mathcal{D}_d^{(0)}$ – New results



Basic concept:

Divide a given convex region inside the interior of $\mathcal{D}_d$ into finitely many classes related to sets of witnesses.

Basic concept:

Divide a given convex region inside the interior of $\mathcal{D}_d$ into finitely many classes related to sets of witnesses.

Each class is either contained in $\mathcal{D}_d^{(0)}$ or has empty intersection with it.

Basic concept:

Divide a given convex region inside the interior of $\mathcal{D}_d$ into finitely many classes related to sets of witnesses.

Each class is either contained in $\mathcal{D}_d^{(0)}$ or has empty intersection with it.

Handle classes in a sophisticated order to minimize computation time!

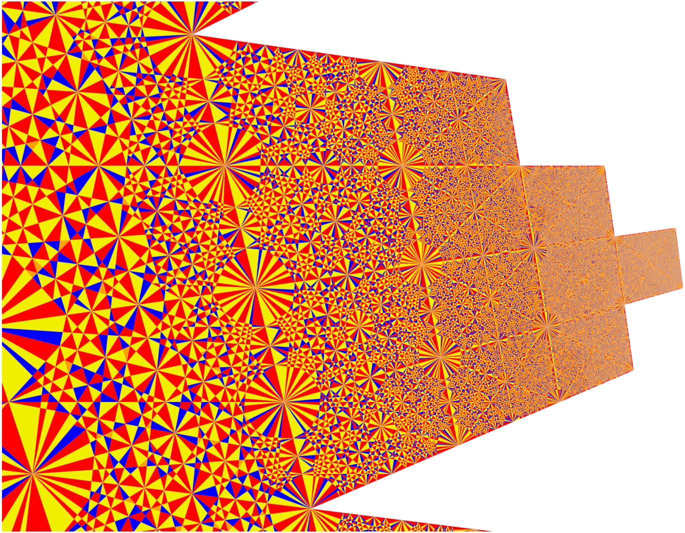


Figure: Hidden structure revealed by classes related to sets of witnesses

Advantages of the first algorithm:

- Faster than Brunotte's algorithm

Advantages of the first algorithm:

- Faster than Brunotte's algorithm
- A "real" algorithm (terminates for all inputs)

Advantages of the first algorithm:

- Faster than Brunotte's algorithm
- A "real" algorithm (terminates for all inputs)

Advantages of the second algorithm:

- Much faster than Brunotte's algorithm

Two new algorithms

Advantages of the first algorithm:

- Faster than Brunotte's algorithm
- A "real" algorithm (terminates for all inputs)

Advantages of the second algorithm:

- Much faster than Brunotte's algorithm
- Very compact output (minimal list of cutout polyhedra)

Thank you for your attention!

Motivation - Relation to Canonical Number Systems

Let
$P(X) = X^d + p_{d-1}X^{d-1} + \cdots + p_1 X + p_0 \in \mathbb{Z}[X]$
$\mathcal{R} := \mathbb{Z}[X]/P(X)\mathbb{Z}[X]$
$\mathcal{N} := \{0, 1, \ldots, |p_0| - 1\}$
$x := X + P(X)\mathbb{Z}[X] \in \mathcal{R}$

Let
$P(X) = X^d + p_{d-1}X^{d-1} + \cdots + p_1 X + p_0 \in \mathbb{Z}[X]$
$\mathcal{R} := \mathbb{Z}[X]/P(X)\mathbb{Z}[X]$
$\mathcal{N} := \{0, 1, \ldots, |p_0| - 1\}$
$x := X + P(X)\mathbb{Z}[X] \in \mathcal{R}$

$(P, \mathcal{N})$ is called a CNS, P a CNS polynomial and $\mathcal{N}$ the set of digits if every non-zero element $A(x) \in \mathcal{R}$ can be represented uniquely in the form

$$A(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$$

with $m \in \mathbb{N}_0$, $a_i \in \mathcal{N}$ and $a_m \neq 0$.

Motivation - Relation to Canonical Number Systems

Let
$P(X) = X^d + p_{d-1}X^{d-1} + \cdots + p_1 X + p_0 \in \mathbb{Z}[X]$
$\mathcal{R} := \mathbb{Z}[X]/P(X)\mathbb{Z}[X]$
$\mathcal{N} := \{0, 1, \ldots, |p_0| - 1\}$
$x := X + P(X)\mathbb{Z}[X] \in \mathcal{R}$

$(P, \mathcal{N})$ is called a CNS, P a CNS polynomial and $\mathcal{N}$ the set of digits if every non-zero element $A(x) \in \mathcal{R}$ can be represented uniquely in the form

$$A(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$$

with $m \in \mathbb{N}_0$, $a_i \in \mathcal{N}$ and $a_m \neq 0$.

Then P is a CNS polynomial $\iff$ $\left(\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \ldots, \frac{p_2}{p_0}, \frac{p_1}{p_0}\right) \in \mathcal{D}_d^{(0)}$