(Gaussian) Shift Radix Systems - recent discoveries and computational results

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In particular for $d = 2 \leftrightarrow d = 1$:

Identify $\mathbb{C} \leftrightarrow \mathbb{R}^2$ and $\mathbb{Z}[\mathrm{i}] \leftrightarrow \mathbb{Z}^2$

$$\begin{aligned}\tau_{(r,s)} : \mathbb{Z}^2 &\mapsto \mathbb{Z}^2 \\ (a,b) &\to (b, -\lfloor ra + sb \rfloor)\end{aligned}$$

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Orbit of $(0,3)$ ultimately periodic!
$\mathbf{r} \in \mathcal{D}_2$?

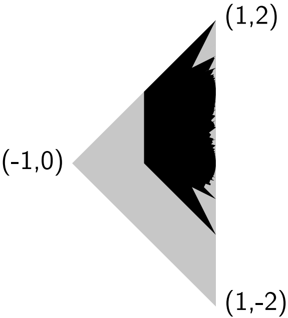


Figure: $\mathcal{D}_2^{(0)}$ in $\mathcal{D}_2$

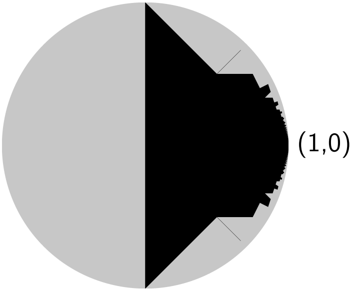
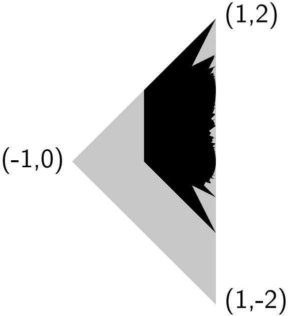


Figure: $\mathcal{G}_1^{(0)}$ in $\mathcal{G}_1$

Interested in $\mathcal{D}_d^{(0)}$ and $\mathcal{G}_d^{(0)}$



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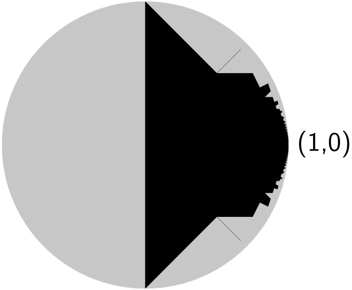

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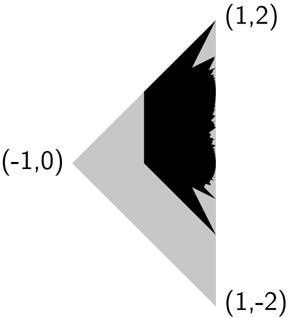

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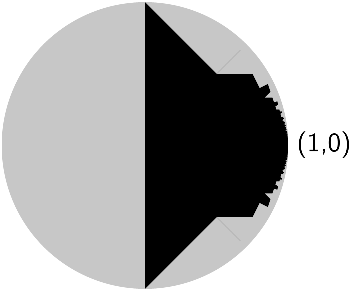

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Relation between SRS, β-Expansions, and Canonical Number Systems

Motivation - Relation to β-Expansions

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Then $\mathcal{A} := \{0, 1, \ldots, \lfloor\beta\rfloor\}$ is called the set of digits, as every $\gamma \in [0, \infty)$ can be represented uniquely in the form

$$\gamma = a_m\beta^m + a_{m-1}\beta^{m-1} + \cdots$$

(greedy expansion of γ with respect to β)

with $m \in \mathbb{Z}$ and $a_i \in \mathcal{A}$, such that

$$0 \leq \gamma - \sum_{i=k}^{m} a_i\beta^i < \beta^k$$

holds for all $k \leq m$.

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Example:

$$\beta = \varphi = \tfrac{1+\sqrt{5}}{2} = 1.6180339887\ldots \; (\Rightarrow \mathcal{A} = \{0,1\})$$
$$\gamma = \tfrac{5}{\varphi} - \tfrac{11}{\varphi^3} = 0.49342219125\ldots$$

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Motivation - Relation to β-Expansions

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Let $P(X) = X^d + p_{d-1}X^{d-1} + \cdots + p_1 X + p_0 \in \mathbb{Z}[X]$

Motivation - Relation to Canonical Number Systems

A similar relation can be shown for CNS:

Let $P(X) = X^d + p_{d-1}X^{d-1} + \cdots + p_1X + p_0 \in \mathbb{Z}[X]$

Then P is a CNS polynomial $\iff (\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \dots, \frac{p_2}{p_0}, \frac{p_1}{p_0}) \in \mathcal{D}_d^{(0)}$
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For $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{R}^d$ let

$$R_{\mathbf{r}} := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -r_1 & -r_2 & \cdots & \cdots & -r_d \end{pmatrix}$$

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Then $\tau_{\mathbf{r}}(\mathbf{x}) = R_{\mathbf{r}}\mathbf{x} + \mathbf{v}_{\mathbf{x}}$ where $\mathbf{v}_{\mathbf{x}} = (0, \dots, 0, \{\mathbf{r}\mathbf{x}\})$.

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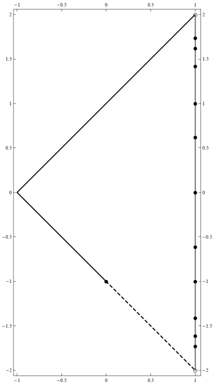

Figure: $\mathcal{D}_2$

Points on right line: $\frac{\pm 1 \pm \sqrt{5}}{2}$, $\pm\sqrt{2}$, $\pm\sqrt{3}$ (quadratic irrational numbers)

Characterization of $\mathcal{D}_d^{(0)}$ and $\mathcal{G}_d^{(0)}$– Two important concepts

Cutout polyhedra

For a tuple π of vectors in $\mathbb{Z}^d$ let $\mathcal{P}(\pi)$ denote the set of all $\mathbf{r} \in \mathbb{R}^d$ for which π is a period of $\tau_\mathbf{r}$.

$\pi = (\mathbf{x_1}, \ldots, \mathbf{x_n})$, $\tau_\mathbf{r}(\mathbf{x_1}) = \mathbf{x_2}$, $\ldots$,$\tau_\mathbf{r}(\mathbf{x_n}) = \mathbf{x_1}$

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Sets of witnesses

A set $V \subseteq \mathbb{Z}^d$ is called a set of witnesses for $\mathbf{r} \in \mathbb{R}^d$ iff it is stable under $\tau_{\mathbf{r}}$ and $\tau_{\mathbf{r}}^{\star} := -\tau_{\mathbf{r}} \circ (-\mathrm{id}_{\mathbb{Z}^d})$ and contains a generating set of the group $(\mathbb{Z}^d, +)$ which is closed under taking inverses.

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Every such set of witnesses has the decisive property:

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Find a finite set of witnesses iteratively for $\mathbf{r} \in \mathrm{int}\,(\mathcal{D}_d)$:

$$V_0 := \{(\pm 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, \pm 1)\}$$
$$\forall\, n \in \mathbb{N} : V_n := V_{n-1} \cup \tau_{\mathbf{r}}(V_{n-1}) \cup \tau_{\mathbf{r}}^\star(V_{n-1})$$
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(BRUNOTTE 2001)

Characterization of $\mathcal{D}_d^{(0)}$ and $\mathcal{G}_d^{(0)}$ – Two important concepts

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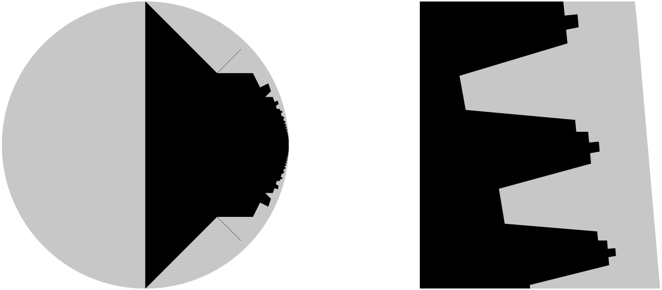
Characterization of $\mathcal{D}_d^{(0)}$ and $\mathcal{G}_d^{(0)}$ – Two important concepts

Example: V for $r = \frac{9}{10} + \mathrm{i}\frac{6}{17}$

Characterization of $\mathcal{D}_d^{(0)}$ and $\mathcal{G}_d^{(0)}$ – Two important concepts

Example: V for $r = \frac{9}{10} + \mathrm{i}\frac{6}{17}$

Characterization of $\mathcal{G}_1^{(0)}$ (Pethő's Loudspeaker)



$\mathcal{G}_1^{(0)}$ is contained in the closed right half of the closed unit disk, and is symmetric with respect to the real axis (BRUNOTTE et al. 2011).

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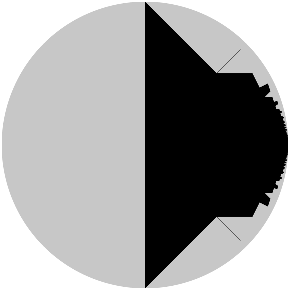

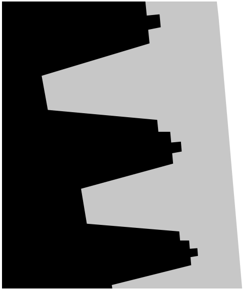

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Conjecture by M.W.:

$\mathcal{G}_1^{(0)} = \mathcal{G}_C$ where $\mathcal{G}_C$ is a (neither open nor closed) polygon given by ten infinite sequences of points in $\mathbb{C}$ (and their complex conjugates).

Ten sequences:

$$z_1(n) = 1 + \frac{-2+\mathrm{i}n}{n^2-2}$$

$$z_2(n) = 1 + \frac{-1+\mathrm{i}(n-1)}{n^2-n-1}$$

$$z_3(n) = 1 + \frac{-1+\mathrm{i}(n-1)}{n^2-n}$$

$$z_4(n) = 1 + \frac{-1+\mathrm{i}n}{n^2}$$

$$z_5(n) = 1 + \frac{-1+\mathrm{i}n}{n^2+1}$$

$$z_6(n) = 1 + \frac{-1+\mathrm{i}(n+1)}{n^2+n+1}$$

$$z_7(n) = 1 + \frac{-1+\mathrm{i}(n+1)}{n^2+n+2}$$

$$z_8(n) = 1 + \frac{-1+\mathrm{i}n}{n^2+2}$$

$$z_9(n) = 1 + \frac{-1+\mathrm{i}n}{n^2+3}$$

$$z_{10}(n) = 1 + \frac{-2+\mathrm{i}(n+1)}{n^2+n+6}$$

Characterization of $\mathcal{G}_1^{(0)}$ (Pethő's Loudspeaker)

Already proven: $\mathcal{G}_1^{(0)} \subseteq \mathcal{G}_C$

Result achieved by identification of 20 families of cutout polygons

Characterization of $\mathcal{G}_1^{(0)}$ (Pethő's Loudspeaker)

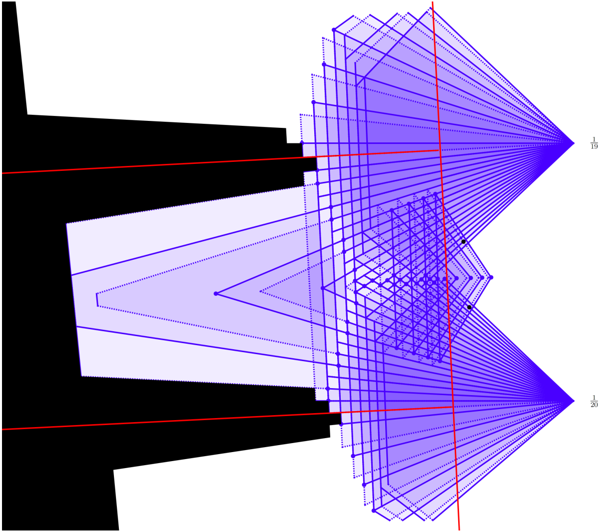


Figure: 20 families of cutout polygons

Characterization of $\mathcal{G}_1^{(0)}$ (Pethő's Loudspeaker)

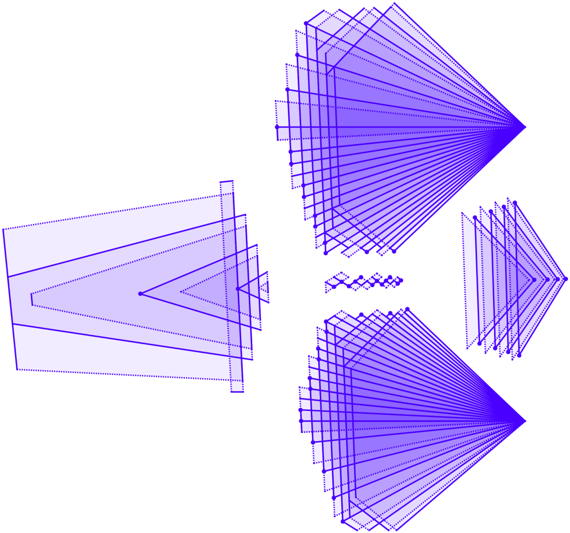


Figure: 20 families of cutout polygons

Characterization of $\mathcal{G}_1^{(0)}$ (Pethő's Loudspeaker)

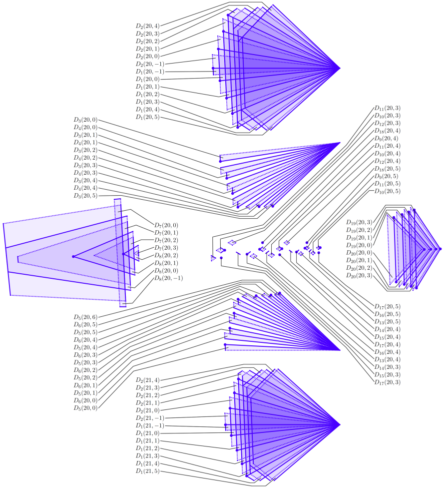


Figure: 20 families of cutout polygons

Characterization of $\mathcal{G}_1^{(0)}$ (Pethő's Loudspeaker)

Other inclusion: Settled inside $\left\{\mathbf{r} \in \mathbb{C} \mid |\mathbf{r}| \le \frac{1023}{1024}\right\}$ by a new algorithm

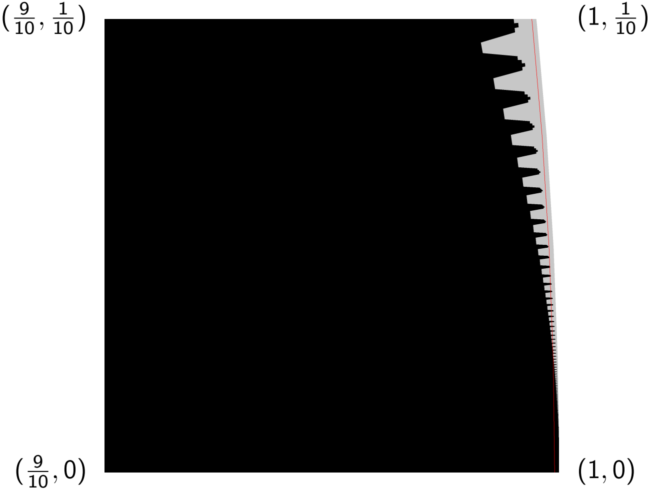


Figure: Region settled algorithmically

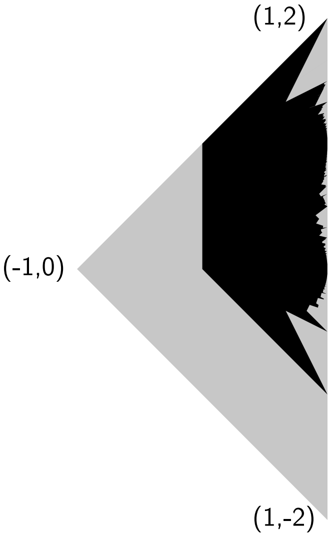
Characterization of $\mathcal{G}_1^{(0)}$ (Pethő's Loudspeaker)

Hope for full characterization of $\mathcal{G}_1^{(0)}$
by thorough investigation of orbits of $\gamma_{\mathbf{r}}$!

- $\mathcal{D}_1 = [-1, 1]$, $\mathcal{D}_1^{(0)} = [0, 1)$

Characterization of $\mathcal{D}_d^{(0)}$ – Previous results

- $\mathcal{D}_1 = [-1, 1]$, $\mathcal{D}_1^{(0)} = [0, 1)$
- $\mathcal{D}_2 \subseteq \{(x, y) \in \mathbb{R}^2 \mid x \geq |y| - 1 \wedge x \leq 1\}$



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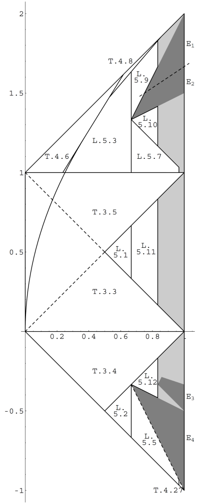

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  Applied by SURER 2008 to characterize $\mathcal{D}_2^{(0)} \cap \left\{(x, y) \in \mathbb{R}^2 \mid x \leq L\right\}$ where $L = \frac{99}{100}$

Characterization of $\mathcal{D}_d^{(0)}$ – New results

Theorem (M.W.):

- $\mathcal{D}_2^{(0)}$ has at least 22 connected components
- The largest connected component of $\mathcal{D}_2^{(0)}$ has at least 3 holes

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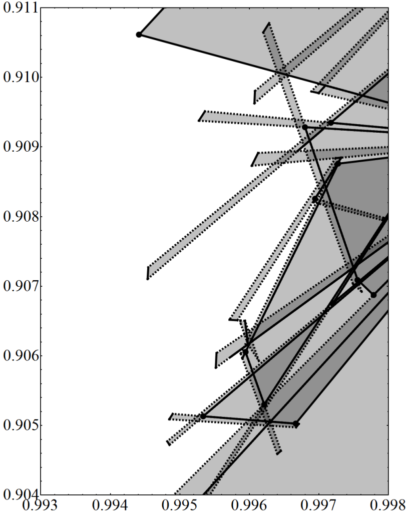
Result achieved by a new algorithm which has been used to characterize $\mathcal{D}_2^{(0)} \cap \{(x, y) \in \mathbb{R}^2 \mid x \leq L\}$ where $L = \frac{511}{512}$.


0.911
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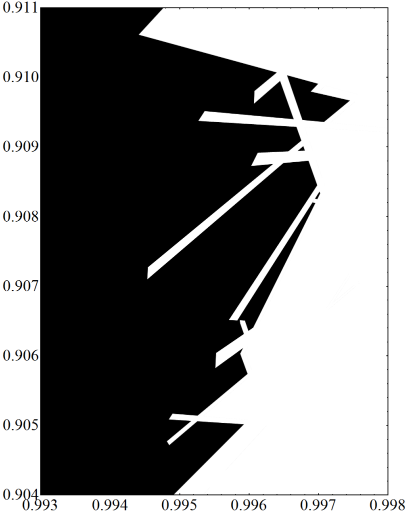
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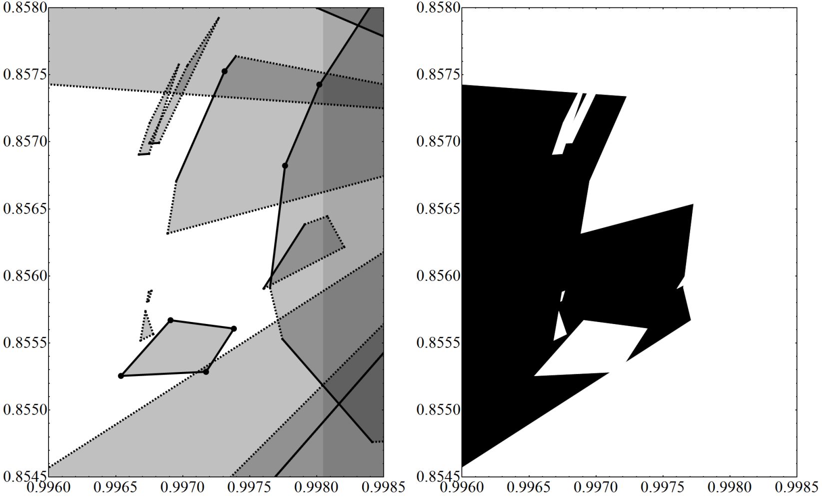
Characterization of $\mathcal{D}_d^{(0)}$ – New results



Basic concept:

Divide a given convex region inside the interior of $\mathcal{D}_d$ into finitely many classes related to sets of witnesses.

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Each class is either contained in $\mathcal{D}_d^{(0)}$ or has empty intersection with it.

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Handle classes in a sophisticated order to minimize computation time!

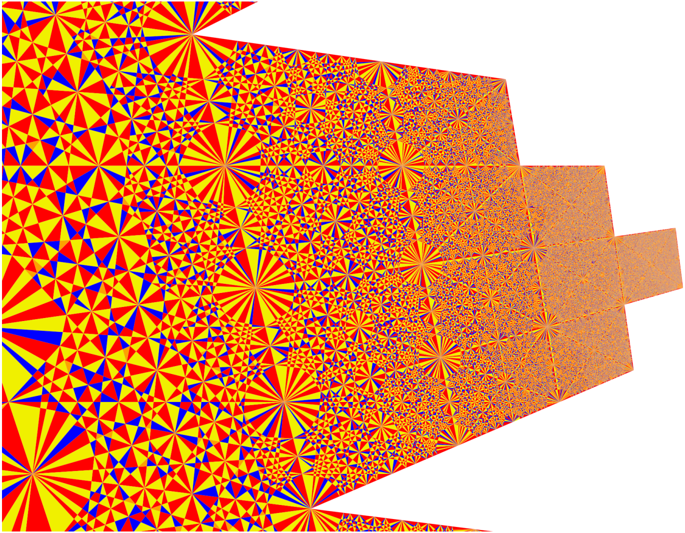
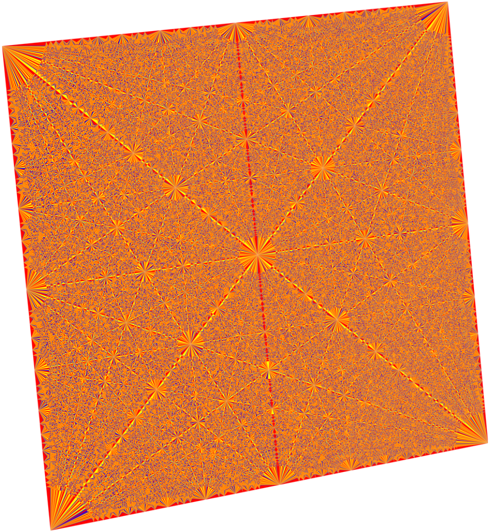
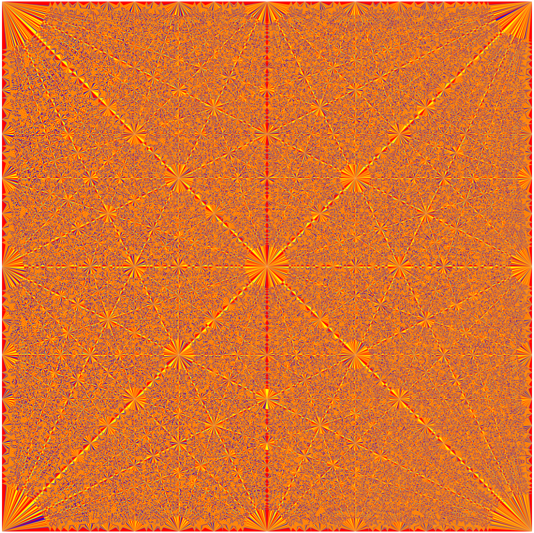
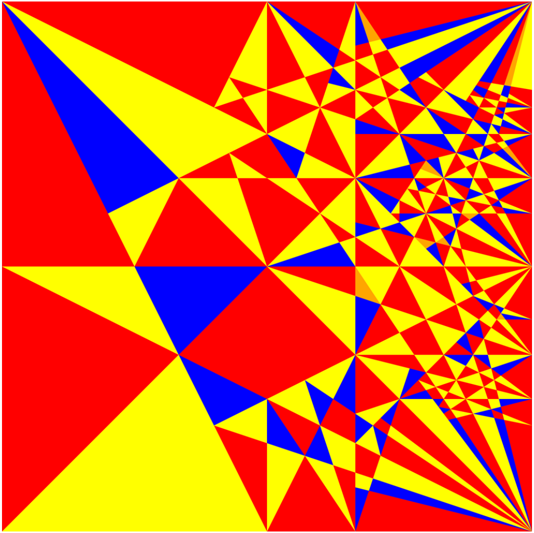


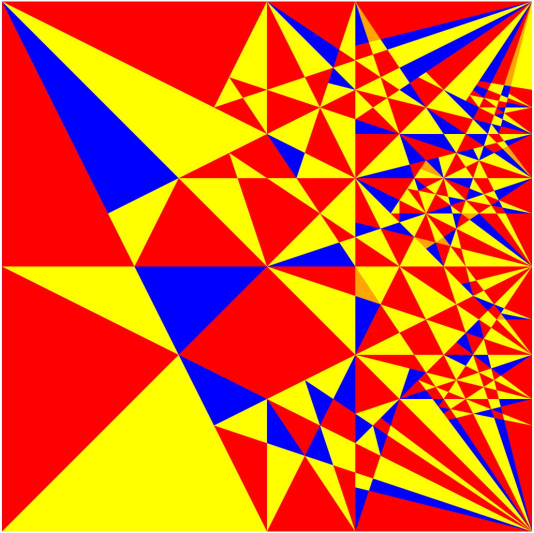
Figure: Hidden structure revealed by classes related to sets of witnesses

Two new algorithms









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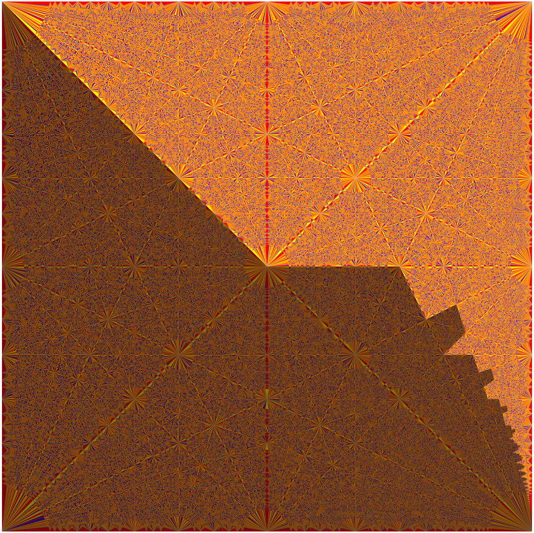

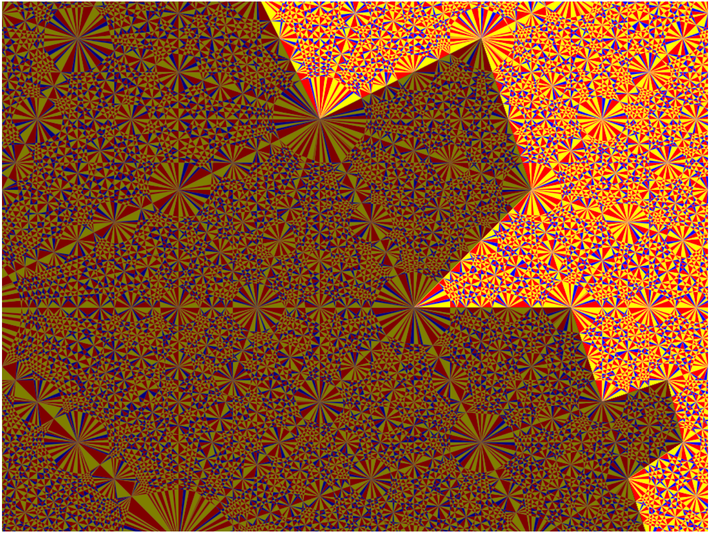

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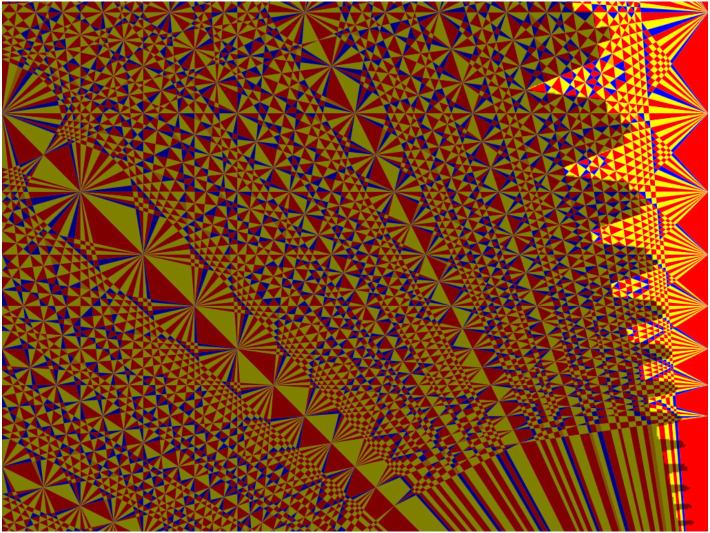

Advantages of the first algorithm:

- Faster than Brunotte's algorithm

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- Faster than Brunotte's algorithm
- A "real" algorithm (terminates for all inputs)

Advantages of the second algorithm:

- Much faster than Brunotte's algorithm
- Very compact output (minimal list of cutout polyhedra)

Thank you for your attention!

Let
$P(X) = X^d + p_{d-1}X^{d-1} + \cdots + p_1X + p_0 \in \mathbb{Z}[X]$
$\mathcal{R} := \mathbb{Z}[X]/P(X)\mathbb{Z}[X]$
$\mathcal{N} := \{0, 1, \ldots, |p_0| - 1\}$
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$(P, \mathcal{N})$ is called a CNS, P a CNS polynomial and $\mathcal{N}$ the set of digits if every non-zero element $A(x) \in \mathcal{R}$ can be represented uniquely in the form

$$A(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$$

with $m \in \mathbb{N}_0$, $a_i \in \mathcal{N}$ and $a_m \neq 0$.

Motivation - Relation to Canonical Number Systems

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Then P is a CNS polynomial $\iff (\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \ldots, \frac{p_2}{p_0}, \frac{p_1}{p_0}) \in \mathcal{D}_d^{(0)}$