

(Gaussian) Shift Radix Systems - recent discoveries and computational results

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$$\mathbf{x} = (x_1, \dots, x_d) \rightarrow (x_2, \dots, x_d, -\lfloor \mathbf{r}\mathbf{x} \rfloor)$$

is called the d - dimensional **SRS** associated with \mathbf{r} (AKIYAMA et al. 2005)

where $\mathbf{r}\mathbf{x} = \sum_{i=1}^d r_i x_i$ denotes the scalar product of \mathbf{r} and \mathbf{x}
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Relation between SRS and GSRS

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In particular for $d = 2 \leftrightarrow d = 1$:

Identify $\mathbb{C} \leftrightarrow \mathbb{R}^2$ and $\mathbb{Z}[i] \leftrightarrow \mathbb{Z}^2$

$$\tau_{(r,s)} : \mathbb{Z}^2 \mapsto \mathbb{Z}^2$$

$$(a, b) \mapsto (b, -\lfloor ra + sb \rfloor)$$

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$$(a, b) \mapsto (-\lfloor ra - sb \rfloor, -\lfloor rb + sa \rfloor)$$

Example:

$$d = 2$$

$$\mathbf{r} = \left(\frac{9}{10}, \frac{13}{10}\right) \in \mathbb{R}^2$$

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Orbit of $(0, 3)$ ultimately periodic!

$$\mathbf{r} \in \mathcal{D}_2?$$

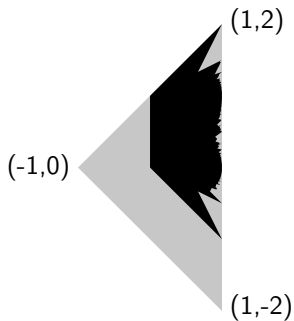


Figure: $\mathcal{D}_2^{(0)}$ in \mathcal{D}_2

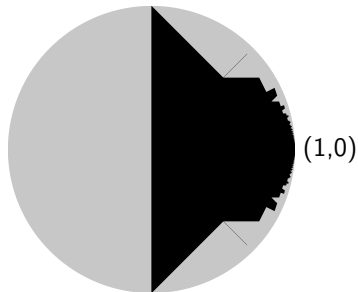


Figure: $\mathcal{G}_1^{(0)}$ in \mathcal{G}_1

Interested in $\mathcal{D}_d^{(0)}$ and $\mathcal{G}_d^{(0)}$

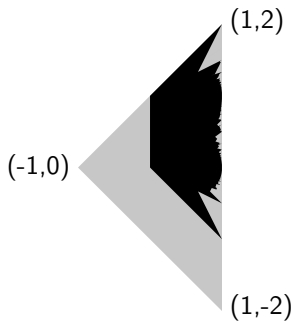


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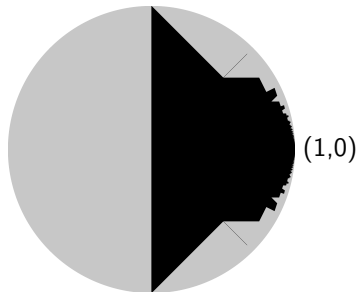


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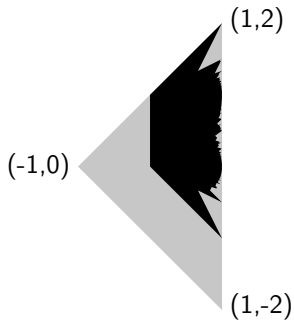


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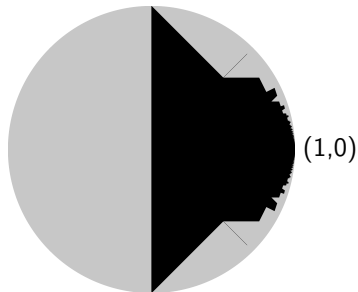


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Relation between SRS, β -Expansions, and Canonical Number Systems

Motivation - Relation to β -Expansions

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Then $\mathcal{A} := \{0, 1, \dots, \lfloor \beta \rfloor\}$ is called the **set of digits**, as every $\gamma \in [0, \infty)$ can be represented uniquely in the form

$$\gamma = a_m \beta^m + a_{m-1} \beta^{m-1} + \dots$$

(**greedy expansion** of γ with respect to β)

with $m \in \mathbb{Z}$ and $a_i \in \mathcal{A}$, such that

$$0 \leq \gamma - \sum_{i=k}^m a_i \beta^i < \beta^k$$

holds for all $k \leq m$.

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Example:

$$\begin{aligned}\beta &= \varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887 \dots (\Rightarrow \mathcal{A} = \{0, 1\}) \\ \gamma &= \frac{5}{\varphi} - \frac{11}{\varphi^3} = 0.49342219125 \dots\end{aligned}$$

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$$0.493 \dots (0) \xrightarrow{T_\beta} 0.798 \dots (1)$$

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Then β has **property (F)** $\iff (r_0, \dots, r_{d-2}) \in \mathcal{D}_{d-1}^{(0)}$ (AKIYAMA et al. 2005)

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For $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{R}^d$ let

$$R_{\mathbf{r}} := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -r_1 & -r_2 & \cdots & \cdots & -r_d \end{pmatrix}$$

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Equivalent statements are true for \mathcal{G}_d .

The boundary of \mathcal{D}_2

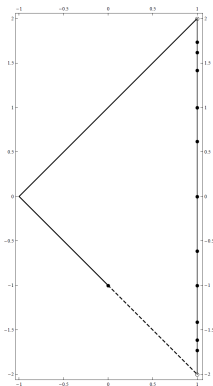


Figure: \mathcal{D}_2

Points on right line: $\frac{\pm 1 \pm \sqrt{5}}{2}$, $\pm\sqrt{2}$, $\pm\sqrt{3}$ (quadratic irrational numbers)

Characterization of $\mathcal{D}_d^{(0)}$ and $\mathcal{G}_d^{(0)}$ – Two important concepts

Cutout polyhedra

For a tuple π of vectors in \mathbb{Z}^d let $\mathcal{P}(\pi)$ denote the set of all $\mathbf{r} \in \mathbb{R}^d$ for which π is a period of $\tau_{\mathbf{r}}$.

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A set $V \subseteq \mathbb{Z}^d$ is called a **set of witnesses** for $\mathbf{r} \in \mathbb{R}^d$ iff it is stable under $\tau_{\mathbf{r}}$ and $\tau_{\mathbf{r}}^* := -\tau_{\mathbf{r}} \circ (-\text{id}_{\mathbb{Z}^d})$ and contains a generating set of the group $(\mathbb{Z}^d, +)$ which is closed under taking inverses.

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Find a **finite** set of witnesses iteratively for $\mathbf{r} \in \text{int}(\mathcal{D}_d)$:

$$\begin{aligned} V_0 &:= \{(\pm 1, 0, \dots, 0), \dots, (0, \dots, 0, \pm 1)\} \\ \forall n \in \mathbb{N} : V_n &:= V_{n-1} \cup \tau_{\mathbf{r}}(V_{n-1}) \cup \tau_{\mathbf{r}}^*(V_{n-1}) \\ V_{\mathbf{r}} &:= \bigcup_{n \in \mathbb{N}_0} V_n \end{aligned}$$

(BRUNOTTE 2001)

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A set $V \subseteq \mathbb{Z}^d$ is called a **set of witnesses** for $\mathbf{r} \in \mathbb{R}^d$ iff it is stable under $\tau_{\mathbf{r}}$ and $\tau_{\mathbf{r}}^* := -\tau_{\mathbf{r}} \circ (-\text{id}_{\mathbb{Z}^d})$ and contains a generating set of the group $(\mathbb{Z}^d, +)$ which is closed under taking inverses.

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$$\mathbf{r} \in \mathcal{D}_d^{(0)} \Leftrightarrow \forall \mathbf{a} \in V : \exists n \in \mathbb{N} : \tau_{\mathbf{r}}^n(\mathbf{a}) = \mathbf{0}$$

Find a **finite** set of witnesses iteratively for $\mathbf{r} \in \text{int}(\mathcal{D}_d)$:

$$\begin{aligned} V_0 &:= \{(\pm 1, 0, \dots, 0), \dots, (0, \dots, 0, \pm 1)\} \\ \forall n \in \mathbb{N} : V_n &:= V_{n-1} \cup \tau_{\mathbf{r}}(V_{n-1}) \cup \tau_{\mathbf{r}}^*(V_{n-1}) \\ V_{\mathbf{r}} &:= \bigcup_{n \in \mathbb{N}_0} V_n \end{aligned}$$

(BRUNOTTE 2001)

Equivalent statements are **true for \mathcal{G}_d and $\mathcal{G}_d^{(0)}$** .

Characterization of $\mathcal{D}_d^{(0)}$ and $\mathcal{G}_d^{(0)}$ – Two important concepts

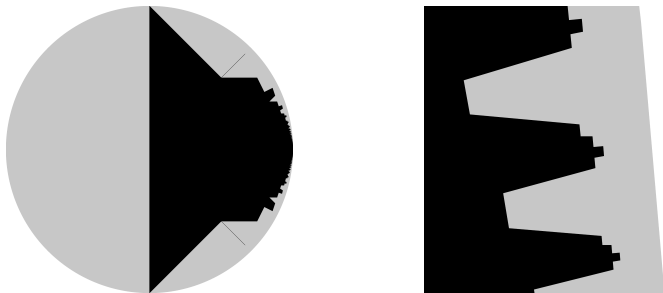
Example: V for $r = \frac{9}{10} + i\frac{6}{17}$



Characterization of $\mathcal{D}_d^{(0)}$ and $\mathcal{G}_d^{(0)}$ – Two important concepts

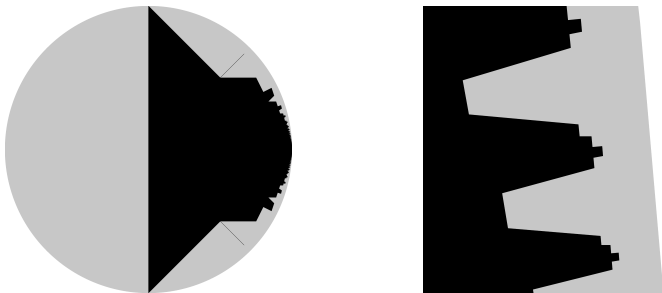
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Characterization of $\mathcal{G}_1^{(0)}$ (Pethő's Loudspeaker)



$\mathcal{G}_1^{(0)}$ is contained in the closed right half of the closed unit disk, and is symmetric with respect to the real axis (BRUNOTTE et al. 2011).

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Conjecture by M.W.:

$\mathcal{G}_1^{(0)} = \mathcal{G}_C$ where \mathcal{G}_C is a (neither open nor closed) polygon given by ten infinite sequences of points in \mathbb{C} (and their complex conjugates).

Characterization of $\mathcal{G}_1^{(0)}$ (Pethő's Loudspeaker)

Ten sequences:

$$z_1(n) = 1 + \frac{-2+in}{n^2-2}$$

$$z_6(n) = 1 + \frac{-1+i(n+1)}{n^2+n+1}$$

$$z_2(n) = 1 + \frac{-1+i(n-1)}{n^2-n-1}$$

$$z_7(n) = 1 + \frac{-1+i(n+1)}{n^2+n+2}$$

$$z_3(n) = 1 + \frac{-1+i(n-1)}{n^2-n}$$

$$z_8(n) = 1 + \frac{-1+in}{n^2+2}$$

$$z_4(n) = 1 + \frac{-1+in}{n^2}$$

$$z_9(n) = 1 + \frac{-1+in}{n^2+3}$$

$$z_5(n) = 1 + \frac{-1+in}{n^2+1}$$

$$z_{10}(n) = 1 + \frac{-2+i(n+1)}{n^2+n+6}$$

Characterization of $\mathcal{G}_1^{(0)}$ (Pethő's Loudspeaker)

Already proven: $\mathcal{G}_1^{(0)} \subseteq \mathcal{G}_c$

Result achieved by identification of 20 families of cutout polygons

Characterization of $\mathcal{G}_1^{(0)}$ (Pethő's Loudspeaker)

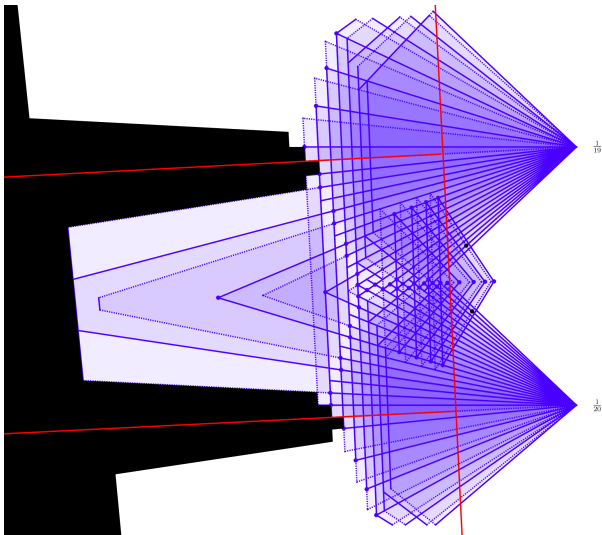


Figure: 20 families of cutout polygons

Characterization of $\mathcal{G}_1^{(0)}$ (Pethő's Loudspeaker)

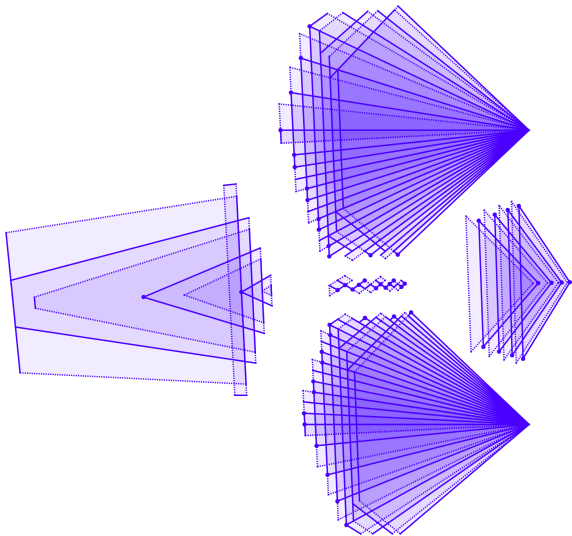


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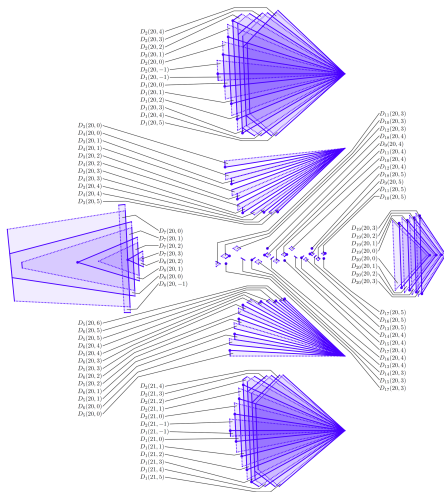


Figure: 20 families of cutout polygons

Characterization of $\mathcal{G}_1^{(0)}$ (Pethő's Loudspeaker)

Other inclusion: Settled inside $\{\mathbf{r} \in \mathbb{C} \mid |\mathbf{r}| \leq \frac{1023}{1024}\}$ by a new algorithm

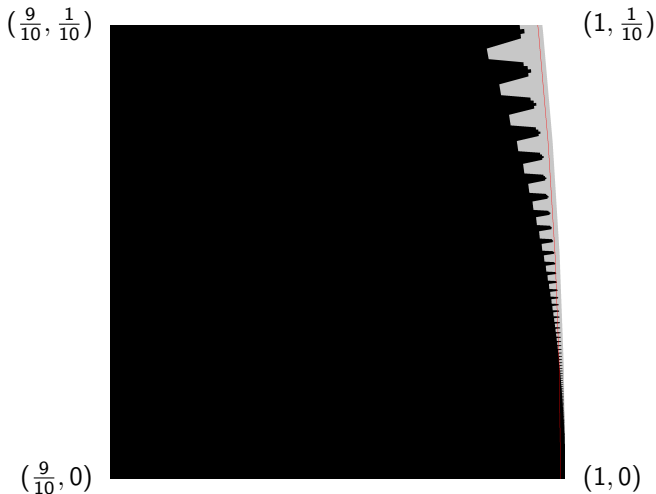


Figure: Region settled algorithmically

Characterization of $\mathcal{G}_1^{(0)}$ (Pethő's Loudspeaker)

Hope for full characterization of $\mathcal{G}_1^{(0)}$
by thorough investigation of orbits of γ_r !

Characterization of $\mathcal{D}_d^{(0)}$ – Previous results

- $\mathcal{D}_1 = [-1, 1]$, $\mathcal{D}_1^{(0)} = [0, 1)$

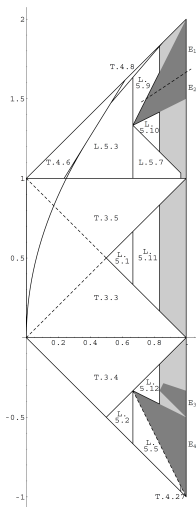
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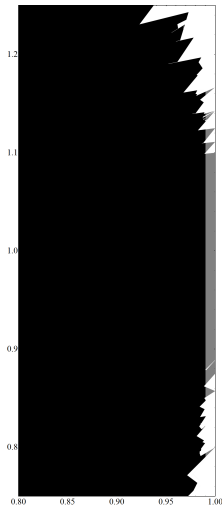
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Applied by SURER 2008 to characterize $\mathcal{D}_2^{(0)} \cap \{(x, y) \in \mathbb{R}^2 \mid x \leq L\}$ where $L = \frac{99}{100}$



Characterization of $\mathcal{D}_d^{(0)}$ – New results

Theorem (M.W.):

- $\mathcal{D}_2^{(0)}$ has at least 22 connected components
- The largest connected component of $\mathcal{D}_2^{(0)}$ has at least 3 holes

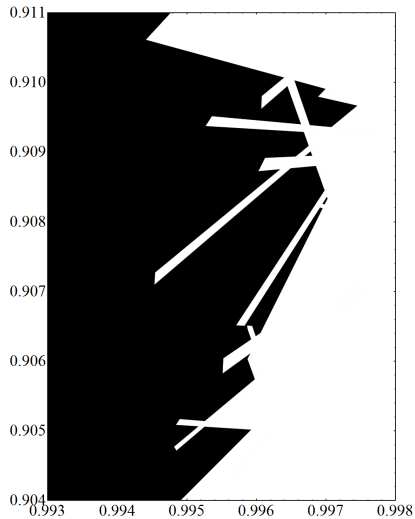
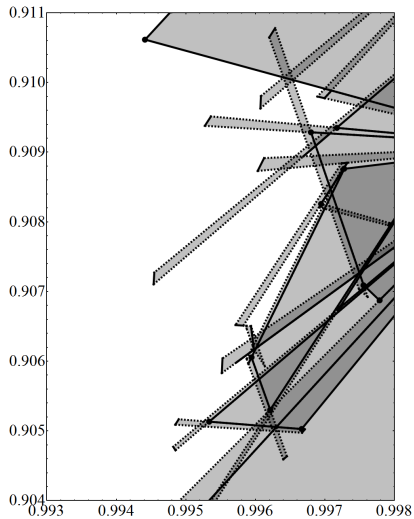
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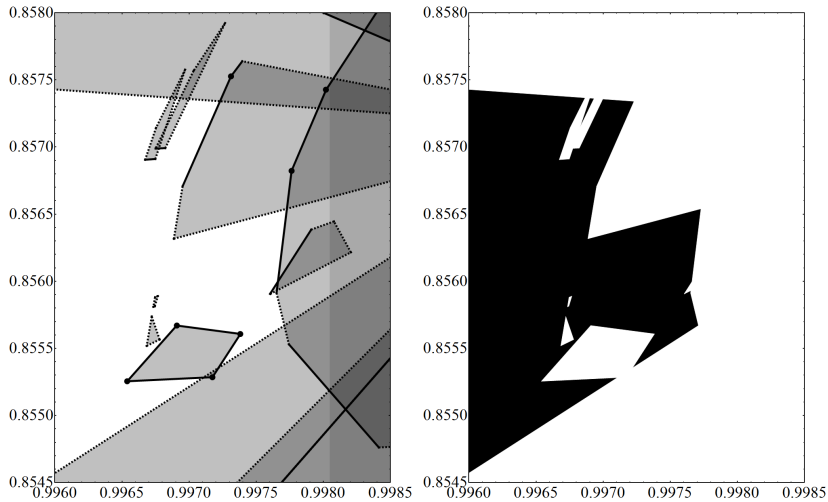
- $\mathcal{D}_2^{(0)}$ has at least 22 connected components
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Result achieved by a new algorithm which has been used to characterize $\mathcal{D}_2^{(0)} \cap \{(x, y) \in \mathbb{R}^2 \mid x \leq L\}$ where $L = \frac{511}{512}$.

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Handle classes in a sophisticated order to minimize computation time!

Two new algorithms

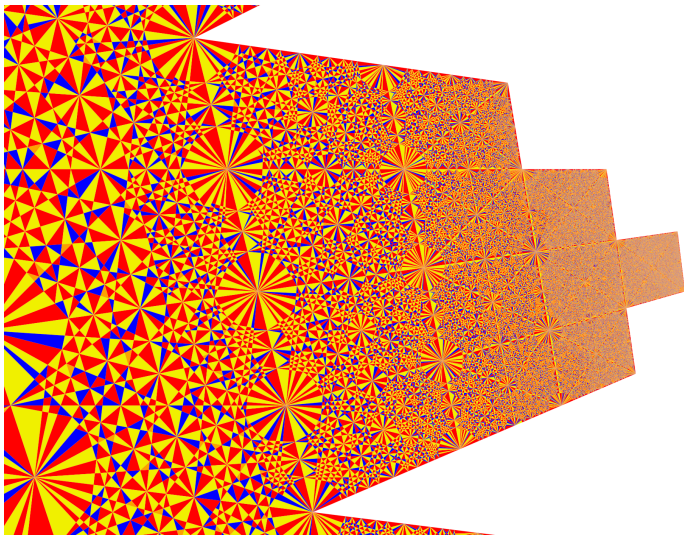
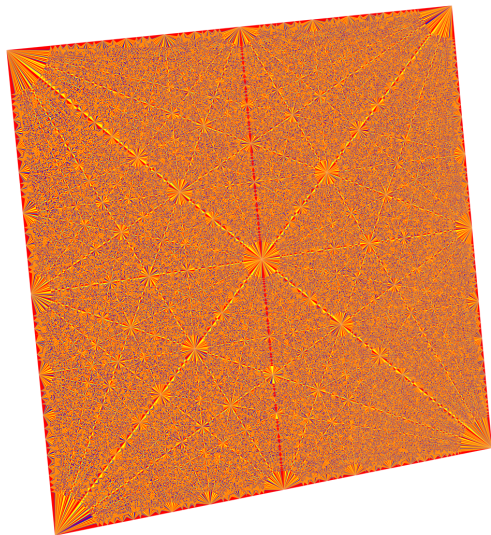
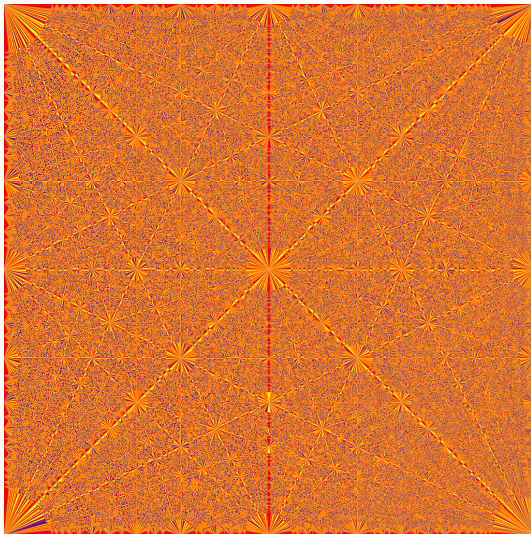


Figure: Hidden structure revealed by classes related to sets of witnesses

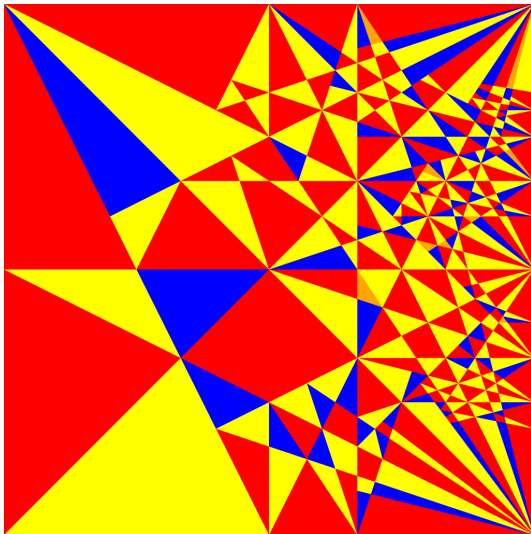
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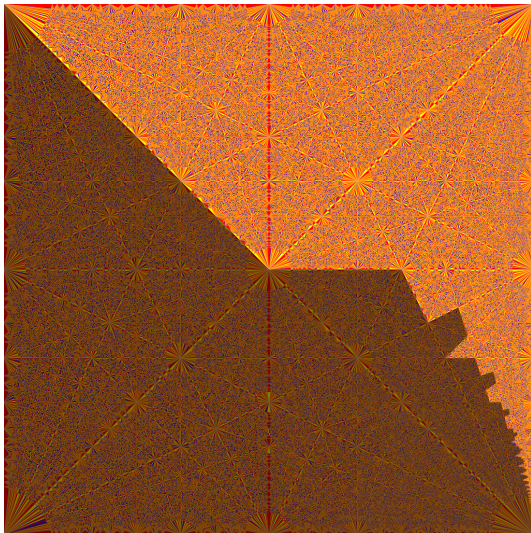


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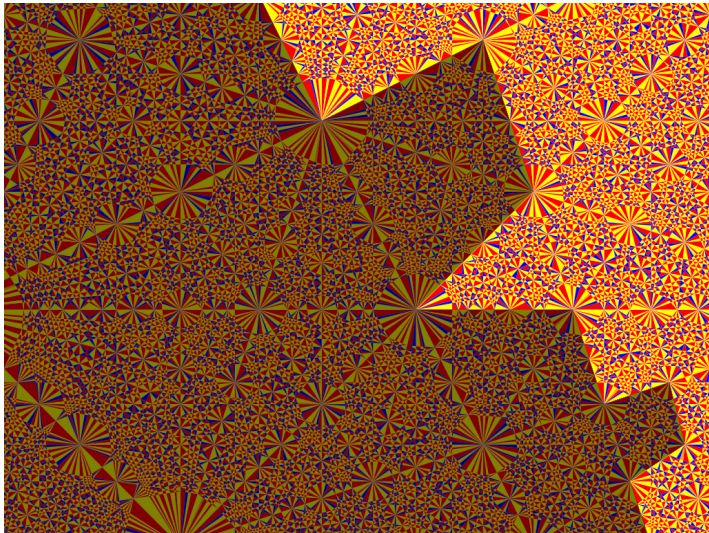


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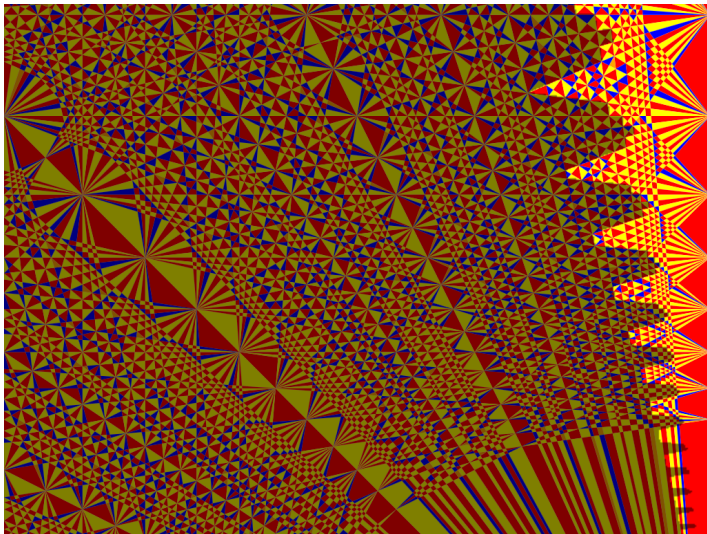
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- A “real” algorithm (terminates for all inputs)

Advantages of the second algorithm:

- Much faster than Brunotte's algorithm
- Very compact output (minimal list of cutout polyhedra)

Thank you for your attention!

Motivation - Relation to Canonical Number Systems

Let

$$P(X) = X^d + p_{d-1}X^{d-1} + \dots + p_1X + p_0 \in \mathbb{Z}[X]$$

$$\mathcal{R} := \mathbb{Z}[X]/P(X)\mathbb{Z}[X]$$

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(P, \mathcal{N}) is called a **CNS**, P a **CNS polynomial** and \mathcal{N} the **set of digits** if every non-zero element $A(x) \in \mathcal{R}$ can be represented uniquely in the form

$$A(x) = a_mx^m + a_{m-1}x^{m-1} + \cdots + a_1x + a_0$$

with $m \in \mathbb{N}_0$, $a_i \in \mathcal{N}$ and $a_m \neq 0$.

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Then P is a **CNS polynomial** $\iff \left(\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \dots, \frac{p_2}{p_0}, \frac{p_1}{p_0}\right) \in \mathcal{D}_d^{(0)}$